CHAPTER 1

# Diophantine approximation and manifolds

## 1.1. Introduction

Diophantine approximation is a more quantitative and general study of the density of the rationals $\mathbb{Q}$ in the reals $\mathbb{R}$ while a smooth manifold is locally diffeomorphic to Euclidean space. In this chapter, those parts of Diophantine approximation and differential geometry needed are set out. The former is concerned mainly with the inequality

$$\left|\xi - \frac{p}{q}\right| < \varepsilon,$$

where $\xi$ is a real number and $\varepsilon$ is a small positive number depending on the rational $p/q$, and its higher dimensional versions. In the metric theory, solution sets of Diophantine inequalities are considered in terms of Lebesgue measure (a knowledge of this is assumed). Because an exceptional set for which a result is false can be of measure zero, this can lead to theorems, such as Khintchine's theorem below, having a strikingly simple yet general character. Moreover the exceptional sets can in turn be analysed in terms of Hausdorff dimension. The analysis becomes much more difficult when considering points on a manifold in Euclidean space, as the coordinates are functionally related and so dependent.

Much of the material in this book is a further development of Sprindžuk's monograph [210] which, starting with a thorough discussion of Khintchine's theorem and its generalisations, goes on to a systematic account of the emergent theory of metric Diophantine approximation on manifolds. J. W. S. Cassels' tract [59] contains a concise but comprehensive introduction to Diophantine approximation and G. Harman's recent book *Metric number theory* [115] has a wider scope which includes brief accounts of Diophantine approximation on manifolds (Chapter 9) and Hausdorff dimension (Chapter 10).

## 1.2. Diophantine approximation in one dimension

Dirichlet's theorem is fundamental to the theory of Diophantine approximation. The one dimensional form of the theorem states that for each real number $\xi$ and any positive integer $N$, there exists a rational $p/q$ with positive denominator $q \leqslant N$, such that

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{qN}.$$

Since $q \leqslant N$, it immediately follows that

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2}. \tag{1.1}$$

If $p$ and $q$ are not restricted to being coprime, then there are infinitely many solutions, otherwise there are only finitely many solutions when $\xi \in \mathbb{Q}$ (for further details see [59], [114], [201]).

It is convenient to introduce some notation. As usual, $\mathbb{N}$ will denote the positive integers $1, 2, \dots$, $\mathbb{Z}$ will denote the integers, the integer part of the real number $\xi$ is the greatest integer at most $\xi$ and will be denoted by $[\xi]$. The fractional part $\xi - [\xi]$ of $\xi$ is non-negative and is written $\{\xi\}$. A standard and simplifying notation which places the denominator $q$ in the foreground is to write

$$\|\xi\| = \min\{|\xi - r| \colon r \in \mathbb{Z}\} = \min\{\{\xi\}, \{1 - \xi\}\},$$

so that (1.1) becomes $\|q\xi\| < 1/q$. Note that $\|\xi + \xi'\| \leqslant \|\xi\| + \|\xi'\|$ and that $\|r\xi\| \leqslant |r|\|\xi\|$ when $r \in \mathbb{Z}$. The *symmetrised* fractional part of $\xi$ defined by

$$\langle\xi\rangle = \begin{cases} \{\xi\} & \text{when } 0 \leqslant \{\xi\} \leqslant 1/2, \\ \{\xi\} - 1 & \text{otherwise,} \end{cases}$$

lies in $(-1/2, 1/2]$ and satisfies $\|\xi\| = |\langle\xi\rangle|$. Given positive real numbers $a, b$, the Vinogradov notation

$$a \ll b \quad \text{or} \quad b \gg a$$

is used for $a = O(b)$, *i.e.*, when $a \leqslant Kb$ for some positive constant $K$. If $a \ll b$ and $a \gg b$, $a$ and $b$ are said to be *comparable*, denoted by $a \asymp b$.

**1.2.1. Approximation functions.** More generally, let $\psi \colon \mathbb{N} \to \mathbb{R}^+$ be a positive function ($\mathbb{R}^+ = \{x \in \mathbb{R} \colon x > 0\}$) where $\psi(q) \to 0$ as $q \to \infty$. Let $X \subseteq \mathbb{R}$. We will write $\mathscr{K}(X;\psi)$ for the set of $\xi \in X$ such that the more general inequality

$$\|q\xi\| < \psi(q) \tag{1.2}$$

holds for infinitely many positive integers $q$, *i.e.*,

$$\mathscr{K}(X;\psi) = \{\xi \in X \colon \|q\xi\| < \psi(q) \text{ for infinitely many } q \in \mathbb{N}\}$$

first studied by A. I. Khintchine [134]. Points in $\mathscr{K}(X;\psi)$ will be called $\psi$-*approximable*. When the set $X$ is clear from the context, we will usually omit reference to it and write simply $\mathscr{K}(\psi)$. The function $\psi$ will be called an *approximation function* and will often be taken to be monotonically decreasing (we will usually omit the term monotonically) as well. Note that $\psi(q) \leqslant 1/2$ when $q$ is sufficiently large. We will make much use of the observation that the set $\mathscr{K}(X;\psi)$ and its generalisations are 'lim-sup' sets as

$$\begin{aligned}\mathscr{K}(X;\psi) &= \{\xi \in X : \xi \in B_{\psi(q)}(q) \text{ for infinitely many } q \in \mathbb{N}\}\\ &= \bigcap_{N=1}^{\infty}\bigcup_{q=N}^{\infty} B_{\psi(q)}(q) = \limsup_{N\to\infty} B_{\psi(N)}(N),\end{aligned}$$

where $B_\delta(q) = \{\xi \in X : \|q\xi\| < \delta\}$. In particular they are Borel sets [100].

The expression $\|\xi\|$ is invariant under translation by integers so that given any integer $r$, $\xi + r$ satisfies (1.2) if and only if $\xi$ does. Thus $\mathscr{K}([0,1]+r;\psi) = \mathscr{K}([0,1];\psi)+r$ and

$$\mathscr{K}(\mathbb{R};\psi) = \bigcup_{r\in\mathbb{Z}} \left(\mathscr{K}([0,1];\psi)+r\right).$$

When considering the measure of the set of $\psi$-approximable real numbers, there is of course no loss in generality in considering points in any (proper) interval.

In the important special case when $\psi(q) = q^{-v}$, we write $\mathscr{K}_v(X)$ for $\mathscr{K}(X;\psi)$; thus

$$\mathscr{K}_v(X) = \{\xi \in X : \|q\xi\| < q^{-v} \text{ for infinitely many } q \in \mathbb{N}\}.$$

Dirichlet's theorem implies that $\mathscr{K}_1(\mathbb{R}) = \mathbb{R}$. Points in $\mathscr{K}_v(\mathbb{R})$ are called $v$-*approximable*; there should be no confusion with $\psi$-approximable points. If a point lies in $\mathscr{K}_v(\mathbb{R})$ for some $v > 1$, it is called *very well approximable* [201]. Thus the set of very well approximable points is the union of $v$-approximable points for $v > 1$. The related set $\mathcal{K}_v(X)$ is defined as follows. For each $\xi \in \mathbb{R}$, let

$$\omega(\xi) = \sup\{w \in \mathbb{R} : \xi \in \mathscr{K}_w(\mathbb{R})\}$$

($\omega(\xi) \geqslant 1$ by Dirichlet's theorem). For any set $X \subseteq \mathbb{R}$ and $v \in \mathbb{R}$, write

$$\mathcal{K}_v(X) = \{\xi \in X : \omega(\xi) \geqslant v\}. \tag{1.3}$$

It is readily verified that $\mathscr{K}_v(X) \subseteq \mathcal{K}_v(X) \subseteq \mathscr{K}_{v-\varepsilon}(X)$ for any $\varepsilon > 0$. The nature of the approximation function in $\mathcal{K}_v(X)$ enables one to analyse $\mathcal{K}'_v(X)$, the set of $\xi$ with $\omega(\xi) = v$ (see §3.5.6).

**1.2.2. Badly approximable numbers.** A number $\xi$ is *badly approximable* if there exists a positive constant $K = K(\xi)$ such that

$$\|q\xi\| \geqslant K/q$$

for all $q \in \mathbb{N}$, *i.e.*, if $\|q\xi\| \gg 1/q$ (but see the Notes and [210, p. 67]). The set of badly approximable numbers will be denoted by $\mathfrak{B}$. By Hurwitz' theorem, for each $\xi \in \mathbb{R}$ the inequality

$$\|q\xi\| < \frac{1}{\sqrt{5}q}$$

holds for infinitely many positive integers $q$. However, the constant $1/\sqrt{5}$ cannot be reduced for numbers $\xi$ equivalent to the golden ratio $(\sqrt{5}-1)/2$. These numbers are thus badly approximable (see [59], [114] for further details). Badly approximable numbers are important in applications, particularly in stability questions for certain dynamical systems (see Chapter 7), but they are amply covered in [201] and so will not be discussed in any detail.

**1.2.3. Khintchine's theorem.** The behaviour of the sum $\sum_{q=1}^{\infty}\psi(q)$ gives an almost complete answer to the solubility of the inequality (1.2). First we need some terminology. A set of Lebesgue measure 0 will usually be called *null*; the complement of a null set is of *full measure* and will usually be called *full*. As usual we will say that *almost no* points belong to a set if it is null while if a set is full we say that it contains *almost all* points. The Lebesgue measure of a set $X$ will be denoted by $|X|$.

THEOREM (KHINTCHINE). *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a function. If the sum $\sum_{q=1}^{\infty}\psi(q)$ converges, then $\mathscr{K}(\mathbb{R};\psi)$ is null, while if the sum diverges and $\psi$ is decreasing, $\mathscr{K}(\mathbb{R};\psi)$ is full.*

Proofs can be found in [59, Chapter VII], [115, Chapter 2] and [210]. In the case of convergence, the result essentially follows from the Borel-Cantelli lemma. Since Cantelli pointed out that the total independence of the events was not needed for convergence [64, p. 507], we will refer for brevity to the convergence part of the lemma as Cantelli's lemma. This will now be stated and proved as it will be used repeatedly throughout.

LEMMA 1.1 (CANTELLI). *Let $(\Omega,\mu)$ be a measure space with $\mu(\Omega)$ finite and let $A_j$, $j \in \mathbb{N}$, be a family of measurable sets. Let*

$$A_\infty = \{\omega \in \Omega \colon \omega \in A_j \text{ for infinitely many } j \in \mathbb{N}\}$$

*and suppose the sum $\sum_{j=1}^{\infty}\mu(A_j) < \infty$. Then $\mu(A_\infty) = 0$.*

PROOF. It is readily verified that $A_\infty$ can be written in 'lim-sup' form as

$$A_\infty = \bigcap_{N=1}^{\infty}\bigcup_{j=N}^{\infty} A_j.$$

It follows that for each $N = 1, 2, \ldots$, the family $\{A_j \colon j \geqslant N\}$ is a cover for the set $A_\infty$, so that $A_\infty \subseteq \bigcup_{j=N}^{\infty} A_j$, whence

$$\mu(A_\infty) \leqslant \sum_{j=N}^{\infty}\mu(A_j).$$

But the sum $\sum_{j=1}^{\infty}\mu(A_j)$ converges whence the tail $\sum_{j=N}^{\infty}\mu(A_j)$ of the series can be made arbitrarily small and so $\mu(A_\infty) = 0$. □

For each $N = 1, 2, \ldots$, the family $\{A_j : j \geqslant N\}$ will be called the *natural* cover for $A_\infty$.

To deduce Khintchine's theorem in the case of convergence, we recall that without loss of generality we can restrict ourselves to the set $[0,1]$. Take $\Omega = [0,1]$ and $\mu$ to be Lebesgue measure. Any point in the set $\mathscr{K}([0,1]\,;\psi)$ lies in infinitely many sets $B_{\psi(q)}(q)$, where

$$B_\delta(q) = \{\xi \in [0,1] : \|q\xi\| < \delta\} = \bigcup_{p=0}^{q} \left(\frac{p}{q} - \frac{\delta}{q}, \frac{p}{q} + \frac{\delta}{q}\right) \cap [0,1]$$

and the family $\{B_{\psi(q)}(q) : q \in \mathbb{N}\}$ is a natural cover for $\mathscr{K}([0,1]\,;\psi)$. Each $B_\delta(q)$ is a union of $q+1$ open intervals. By adding up the lengths of the intervals it can be seen that $|B_\delta(q)| \leqslant 2\delta$ (with equality when $\delta \leqslant 1/2$). Hence the sum $\sum_q |B_{\psi(q)}(q)| \leqslant 2\sum_q \psi(q)$ converges and the result follows from Cantelli's lemma. The case of divergence is harder and a monotonicity condition on the approximation function $\psi$ is required (more details are in [59], [115], [210]); a brief discussion of the more general theorem is in §1.3.4 below.

The theorem corresponds to our intuition since if the approximation function $\psi$ is large then there is a better chance of the inequality being satisfied. Since the sum $\sum_{r=1}^{\infty} r^{-v}$ converges for $v > 1$ and diverges otherwise, Khintchine's theorem implies that the sets $\mathscr{K}_v([0,1])$ and $\mathcal{K}_v([0,1])$ are null or full according as $v > 1$ or $v \leqslant 1$ respectively (in the latter case they are both the real line by Dirichlet's theorem). Less obviously, the theorem shows the Lebesgue measure of the set of $\xi \in [0,1]$ such that (1.2) has infinitely many solutions is 1 when $\psi(q) = 1/(q \log q)$ and 0 when $\psi(q) = 1/(q(\log q)^{1+\varepsilon})$ for any positive $\varepsilon$. This 'zero-one' property is a feature of the metric theory and reflects the links with probability and ergodic theory.

The theorem also implies that the set $\mathfrak{B}$ of badly approximable numbers is null. For given any $K > 0$, the sum $\sum_q (K/q)$ diverges and so by Khintchine's theorem the set of real numbers $\xi$ satisfying $\|q\xi\| < K/q$ for infinitely many $q \in \mathbb{N}$ is full. Thus the complementary set $V(K)$ of $\xi$ such that $\|q\xi\| \geqslant K/q$ for all but finitely many $q$ is null and evidently increases as $K$ decreases. From its definition,

$$\mathfrak{B} \subset \bigcup_{K>0} V(K) = \bigcup_{N=1}^{\infty} V(1/N),$$

a countable union of null sets, whence $\mathfrak{B}$ is null.

## 1.3. Approximation in higher dimensions

The inequality (1.2) can be generalised to higher dimensions. To describe these generalisations concisely, we set down some notation. Throughout, $m, n$ and $N$ will be positive integers, $k, p, q$ will be integers and $q$ will usually be taken to be positive. *Integer* vectors in Euclidean space will always be written with a bold font, thus $\mathbf{q} \in \mathbb{Z}^n$. The *height* or supremum norm $|\xi|_\infty$ of the vector $\xi \in \mathbb{R}^n$ will be denoted by $|\xi|$, so that

$$|\xi| = \max\{|\xi_1|, \dots, |\xi_n|\}.$$

To some extent in number theory the height replaces the usual Euclidean norm which will be written $|\xi|_2$. The inner or scalar product of two vectors $\xi$ and $\zeta$ will be written $\xi \cdot \zeta$. The symmetrised fractional part of a vector $\xi = (\xi_1, \dots, \xi_n)$ in $\mathbb{R}^n$ is defined by

$$\langle \xi \rangle = (\langle \xi_1 \rangle, \dots, \langle \xi_n \rangle) \in (-1/2, 1/2]^n \tag{1.4}$$

and should not be confused with the inner product. Note that there is a unique $\mathbf{k}_\xi \in \mathbb{Z}^n$ such that $\xi - \mathbf{k}_\xi = \langle \xi \rangle$.

The system

$$\xi_1 a_{1j} + \cdots + \xi_m a_{mj}, \ 1 \leqslant j \leqslant n,$$

of $n$ real linear forms in $m$ variables $\xi_1, \dots, \xi_m$ will be written more concisely in matrix form as $\xi A$, where $\xi \in \mathbb{R}^m$ and where by juxtaposing the $m$ rows, the $m \times n$ real matrix $A = (a_{ij})$ is regarded as a point in $\mathbb{R}^{mn}$, *i.e.*, the space of $m \times n$ real matrices is identified with $\mathbb{R}^{mn}$. The inequality (1.2) can be generalised to the system of inequalities

$$|\langle q_1 a_{1j} + \cdots + q_m a_{mj} \rangle| = \|q_1 a_{1j} + \cdots + q_m a_{mj}\| < \psi(|\mathbf{q}|), \quad 1 \leqslant j \leqslant n.$$

Using the notation above, this system can be expressed as

$$|\langle \mathbf{q}\, A \rangle| < \psi(|\mathbf{q}|). \tag{1.5}$$

Matrices satisfying this inequality for infinitely many integer vectors $\mathbf{q}$ are called *$\psi$-approximable* [127]. The extension of Dirichlet's theorem to higher dimensions as a system of simultaneous inequalities involving linear forms is now stated.

THEOREM 1.2. *Let $A = (a_{ij})$ be an $m \times n$ real matrix. For each real $N > 1$, there exists an integer vector $\mathbf{q} \in \mathbb{Z}^m$ with $1 \leqslant |\mathbf{q}| < N$ such that*

$$|\langle \mathbf{q} A \rangle| < N^{-m/n}.$$

Proofs using Minkowski's linear forms theorem are in [59, p. 13, Theorem VI]; similar results using box arguments are in [114], [201].

**1.3.1. Khintchine's transference principle.** Simultaneous and dual Diophantine approximation are related by a 'transference' principle in which a solution in one form is related to a solution in the other form (or more accurately, the form associated with the transpose of the matrix of coefficients). This principle enables information about linear form inequalities and simultaneous Diophantine approximation to be interchanged to a certain extent. In particular it links simultaneous Diophantine approximation on the rational normal curve

$$\mathscr{V} = \mathscr{V}^{(n)} = \{(t, t^2, \dots, t^n) : t \in I\} \tag{1.6}$$

over the interval $I$ with the dual form and so with the distribution of small values of integral polynomials $P(t) = a_0 + a_1 t + \cdots + a_n t^n$, $a_0, \ldots, a_n \in \mathbb{Z}$ for $t \in I$ (see §1.4.4).

THEOREM (KHINTCHINE'S TRANSFERENCE PRINCIPLE). *Suppose the coordinates of $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ are irrational and let $\omega(\xi) \geqslant 0$, $\omega'(\xi) \geqslant 0$ be the respective least upper bounds of the real numbers $w$, $w'$ for which the inequalities*

$$\|q_1\xi_1 + \cdots + q_n\xi_n\| \leqslant (\max |q_i|)^{-n-w},$$
$$\max_{1 \leqslant j \leqslant n} \|q\xi_j\| = |\langle q\xi \rangle| \leqslant q^{-(1+w')/n}$$

*have infinitely many integer solutions. Then*

$$\frac{\omega(\xi)}{n^2 + (n-1)\omega(\xi)} \leqslant \omega'(\xi) \leqslant \omega(\xi)$$

*with the obvious interpretation if $\omega(\xi)$ or $\omega'(\xi)$ is infinite.*

For a proof, see [59, Chapter 5, Theorem IV]. This transference principle implies that given any $\varepsilon > 0$, if $\xi = (\xi_1, \ldots, \xi_n)$ satisfies $|\langle \mathbf{q} \cdot \xi \rangle| < |\mathbf{q}|^{-n-\varepsilon}$ for infinitely many $\mathbf{q} \in \mathbb{Z}^n$, then for some $\varepsilon'$ comparable to $\varepsilon$, $|\langle q\xi \rangle| < q^{-(1+\varepsilon')/n}$ for infinitely many $q \in \mathbb{Z}$, and vice versa. Note that the smaller the modulus of $\omega, \omega'$, the more complete is the interchange of information.

**1.3.2. Two forms of Diophantine approximation.** We will be concerned mainly with the two special cases of the general inequality (1.5), namely when $A$ is a $1 \times n$ real matrix or a $n \times 1$ real matrix (in both cases we regard $A$ as a vector in $\mathbb{R}^n$). A natural question is whether in higher dimensions, subsets such as curves or surfaces, enjoy arithmetic approximation properties corresponding to those for real numbers and $\mathbb{R}^n$.

Let $\xi \in \mathbb{R}^n$ and suppose $\psi(q) \leqslant 1/2$ for all sufficiently large $q \in \mathbb{N}$. First, a point $\xi$ satisfying the system of simultaneous inequalities

$$|\langle q\xi \rangle| = \max\{\|q\xi_1\|, \ldots, \|q\xi_n\|\} < \psi(q) \tag{1.7}$$

lies within (in the sup metric) $\psi(q)/q$ of the point $\mathbf{p}/q$, *i.e.*,

$$\xi \in \{x \in \mathbb{R}^n : |x - \mathbf{p}/q| < \psi(q)/q\},$$

where $\mathbf{p} \in q\,\xi + (-\psi(q), \psi(q))^n$ and is unique when $\psi(q) \leqslant 1/2$. Given a set $X$ in $\mathbb{R}^n$ (later $X$ will be taken to be a manifold), the set of $x \in X$ satisfying (1.7) for infinitely many positive integers $q$ will be denoted by

$$\mathscr{S}(X;\psi) = \{\xi \in X : |\langle q\xi \rangle| < \psi(q) \text{ for infinitely many } q \in \mathbb{N}\}. \tag{1.8}$$

Points in $\mathscr{S}(X;\psi)$ will be called *simultaneously $\psi$-approximable.*

Secondly we consider the 'transposed' or 'dual' inequality

$$|\langle \mathbf{q} \cdot \xi \rangle| = \|\mathbf{q} \cdot \xi\| = \|q_1\xi_1 + \cdots + q_n\xi_n\| < \psi(|\mathbf{q}|), \tag{1.9}$$

involving the linear form $\|\mathbf{q}\cdot\xi\|$. Here the point $\xi$ is within (in the Euclidean metric) $\psi(|\mathbf{q}|)/|\mathbf{q}|_2$ of the hyperplane

$$\{x\in\mathbb{R}^n\colon \mathbf{q}\cdot x = p\},$$

where $p\in\mathbf{q}\cdot\xi+(-\psi(|\mathbf{q}|),\psi(|\mathbf{q}|))$ is unique when $\psi(|\mathbf{q}|)\leqslant 1/2$. The set of $\xi\in X$ satisfying (1.9) for infinitely many integer vectors $\mathbf{q}$ will be denoted by $\mathscr{L}(X;\psi)$,

$$\mathscr{L}(X;\psi)=\{\xi\in X\colon |\langle\mathbf{q}\cdot\xi\rangle|<\psi(|\mathbf{q}|)\text{ for infinitely many }\mathbf{q}\in\mathbb{Z}^n\}. \tag{1.10}$$

Simultaneous Diophantine approximation has a historical priority and it is convenient to refer to the last inequality as the *dual* inequality. Thus points in $\mathscr{L}(X;\psi)$ will be called *dually $\psi$-approximable*. When there is no risk of confusion, we will refer just to $\psi$-approximable points. Note that $\mathscr{S}(X;\psi)$ and $\mathscr{L}(X;\psi)$ are lim-sup sets and that when $X\subseteq\mathbb{R}$, $\mathscr{S}(X;\psi)=\mathscr{L}(X;\psi)=\mathscr{K}(X;\psi)$.

In the important case when $\psi(r)=r^{-v}$, we write $\mathscr{S}_v(X)$ for $\mathscr{S}(X;\psi)$ for $\mathscr{L}(X;\psi)$. The sets $\mathscr{S}_v(X)$ and $\mathscr{L}_v(X)$ decrease as $v$ increases and when $X=\mathbb{R}^n$ are null for $v>1/n$ and $v>n$ respectively [59, Chapter 1]. A point in $\mathscr{S}_v(X)$ will be called *simultaneously $v$-approximable* and a point in $\mathscr{L}_v(X)$ will be called *dually $v$-approximable*. By Khintchine's transference principle, if a point in $\mathbb{R}^n$ is simultaneously $v$-approximable for $v>1/n$, then it is dually $v'$-approximable for some $v'>n$, and vice versa. When $v$ is close to $1/n$, $v'$ is close to $n$ but, as $v$ gets larger, $v'$ gets much further from $n$ and indeed

$$\bigcup_{v>1/n}\mathscr{S}_v(X)=\bigcup_{v>n}\mathscr{L}_v(X). \tag{1.11}$$

Points in this set are called *very well approximable* and the set of such points in $X$ is a countable union of the sets $\mathscr{S}_{1/n+1/r}(X)$ or $\mathscr{L}_{n+1/r}(X)$, $r=1,2,\dots$.

Dirichlet's theorem in higher dimensions specialises to simultaneous Diophantine approximation and to the 'dual' linear form.

Corollary 1.3. *For each $\xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$ there exist*
*(a) an integer $q$ with $1\leqslant q\leqslant N$ and a vector $\mathbf{p}\in\mathbb{Z}^n$ such that*

$$|q\,\xi-\mathbf{p}|<N^{-1/n} \tag{1.12}$$

*and there are infinitely many positive integers $q$ such that*

$$|\langle q\,\xi\rangle|<q^{-1/n},$$

*(b) an integer vector $\mathbf{q}\in\mathbb{Z}^n$ with $1\leqslant|\mathbf{q}|\leqslant N$ and an integer $p$ such that*

$$|\mathbf{q}\cdot\xi-p|<N^{-n}$$

*and there are infinitely many $\mathbf{q}\in\mathbb{Z}^n$ such that*

$$\|\mathbf{q}\cdot\xi\|<|\mathbf{q}|^{-n}.$$

Hence any point in $\mathbb{R}^n$ is simultaneously $(1/n)$-approximable and dually $n$-approximable, so that for any $X \subseteq \mathbb{R}^n$, $\mathscr{S}_{1/n}(X) = \mathscr{L}_n(X) = X$.

The notion of a badly approximable number can also be extended to Euclidean space. The point $\xi \in \mathbb{R}^n$ is *badly approximable* if there exists a positive number $K$ such that

$$|\langle q\xi \rangle| \geqslant K q^{-1/n}$$

for all $q \in \mathbb{N}$; or equivalently by Khintchine's transference principle if there exists a $K' > 0$ such that

$$\|\mathbf{q} \cdot \xi\| \geqslant K' |\mathbf{q}|^{-n}$$

for all non-zero $\mathbf{q} \in \mathbb{Z}^n$. The set of badly approximable points in Euclidean space is null [59, Chapter 1].

The Diophantine approximation considered so far has been homogeneous. The rather different inhomogeneous approximation where for example given $\alpha \in \mathbb{R}$, one considers the inequality $|q\xi - p - \alpha| < \psi(q)$, will not be covered but some further details and references are in the Notes at the end of this chapter and of Chapter 3.

**1.3.3. Order and exponents of approximation.** A real number $\xi$ which for some $K > 0$ satisfies the inequality

$$\left| \xi - \frac{p}{q} \right| < \frac{K}{q^n}$$

for infinitely many rationals $p/q$ is called *(rationally) approximable* to order $n$ [114, §11.2]. Thus if $\xi$ is rationally approximable to order $r + 1$, the inequality

$$\|q\xi\| < K\, q^{-r}$$

holds for infinitely many positive integers $q$. By Dirichlet's theorem every real number can be approximated to order 2. This definition extends naturally to any real exponent and to simultaneous Diophantine approximation and the dual form.

Information about points with exponent of approximation $v$ can be obtained when the approximation function $\psi(q)$ is a power of the form $q^{-v}$. Given a subset $X$ of $\mathbb{R}^n$ and a point $\xi \in X$, let

$$\omega_{\mathscr{S}}(\xi) = \sup\{w \in \mathbb{R} \colon \xi \in \mathscr{S}_w(X)\} \tag{1.13}$$

$$\omega_{\mathscr{L}}(\xi) = \sup\{w \in \mathbb{R} \colon \xi \in \mathscr{L}_w(X)\}. \tag{1.14}$$

Note that if $X = \mathbb{R}^n$, then by Dirichlet's theorem, $\omega_{\mathscr{S}}(\xi) \geqslant 1/n$ and $\omega_{\mathscr{L}}(\xi) \geqslant n$. When $X \subseteq \mathbb{R}^n$ and each real $v$, the set

$$\mathcal{S}_v(X) = \{\xi \in X : \omega_{\mathscr{S}}(\xi) \geqslant v\}$$

$$\mathcal{L}_v(X) = \{\xi \in X : \omega_{\mathscr{S}}(\xi) \geqslant v\}$$

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The sets $\mathcal{S}'_v(X)$ where $\omega_{\mathscr{S}}(\xi) = v$ and $\omega_{\mathscr{L}}(\xi) = v$ respectively in the definitions above, are higher dimensional versions of $\mathcal{K}'_v$.

Also $\mathcal{S}_{1/n}(X) = \mathcal{L}_n(X) = X$ and it can be verified readily that for any $\varepsilon > 0$,

$$\mathscr{S}_v(X) \subseteq \mathcal{S}_v(X) \subseteq \mathscr{S}_{v-\varepsilon}(X) \text{ and } \mathscr{L}_v(X) \subseteq \mathcal{L}_v(X) \subseteq \mathscr{L}_{v-\varepsilon}(X). \quad (1.15)$$

When $n = 1$, $\mathcal{S}_v(\mathbb{R}) = \mathcal{L}_v(\mathbb{R}) = \mathcal{K}_v(\mathbb{R})$.

In order to have the same definition for both types of approximation, we say that a point $\xi \in \mathbb{R}^n$ is *simultaneously approximable to exponent* $v$ if there exists a constant $K$ such that

$$|\langle q\,\xi\rangle| < Kq^{-v}$$

for infinitely many positive integers $q$; and is *dually approximable to exponent* $v$ if there exists a $K'$ such that

$$\|\mathbf{q}\cdot\xi\| < K'|\mathbf{q}|^{-v}$$

for infinitely many $\mathbf{q} \in \mathbb{Z}^n$.

It follows from Khintchine's Transference Principle that for any $X \subseteq \mathbb{R}^n$,

$$|\mathcal{L}_{nv/(1-(n-1)v)}(X)| \leqslant |\mathcal{S}_v(X)| < |\mathcal{L}_{n(1+v)-1}(X)|.$$

The order of approximation can be made more precise. The set

$$\mathcal{S}'_v(X) = \{\xi \in \mathcal{S}(X) \colon \omega_{\mathscr{S}}(\xi) = v\}, \quad (1.16)$$

where $\omega_{\mathscr{S}}(x)$ is given by (1.13), is the set of points in $X$ which can be approximated simultaneously with exact exponent $v$ (but order $v+1$). Similarly, for each $v \in \mathbb{R}$, let

$$\mathcal{L}'_v(X) = \{\xi \in \mathcal{L}(X) \colon \omega_{\mathscr{L}}(\xi) = v\}, \quad (1.17)$$

where $\omega_{\mathscr{L}}(x)$ is given by (1.14). Then $\mathcal{L}'_v(X)$ is the set of points in $X$ which can be approximated dually with exact exponent $v$. The relationship between $\mathscr{S}_v$, $\mathcal{S}_v$ and $\mathcal{S}'_v$ and between $\mathscr{L}_v$, $\mathcal{L}_v$ and $\mathcal{L}'_v$ will be treated further in Chapter 3. The nature of the sets $\mathcal{S}'_v(X)$ and $\mathcal{L}'_v(X)$ allows their Hausdorff dimension to be determined exactly (this is discussed in §3.5.6).

**1.3.4. The Khintchine-Groshev theorem.** A very general form of Khintchine's theorem was obtained by A. V. Groshev (see [210, Chapter 1,§5]). As in the one dimensional case, this result gives precise information about the Lebesgue measure of the set $W(X;\psi)$ of $\psi$-approximable points in the set $X$ when $X$ is Euclidean space or a hypercube (when $X$ is clear, we write simply $W(\psi)$). Further details are given in [83], [210].

THEOREM (GROSHEV). *Let $\psi$ be a function from $\mathbb{N} \to \mathbb{R}^+$. Suppose the sum*

$$\sum_{r=1}^{\infty} r^{m-1}\psi(r)^n \quad (1.18)$$

*converges. Then almost no points $A \in \mathbb{R}^{mn}$ satisfy*

$$|\langle \mathbf{q}A\rangle| < \psi(|\mathbf{q}|) \quad (1.19)$$