1 Truss Structures

1.1 Introduction

A truss structure, also called a *pin-jointed structure*, is a structure consisting of straight, slender members, known as *bars*, connected by frictionless spherical joints. The joints allow each bar to swivel and twist, unless these motions are restricted by connections to other bars. Joints of this kind are an abstraction, as real connections are usually rigidly jointed, but in fact many rigidly jointed structures can be usefully modeled as pin-jointed. Changing the end conditions of the bars in this way has little effect on the overall response of the structure, provided that the pin-jointed version of the actual structure is *kinematically determinate*. This concept will be explained in Section 1.2.

Consider a structure whose straight members are made from a high-modulus material and are rigidly connected to one another. Imagine releasing all of the rotational degrees of freedom at the connections. This change will have little effect on the overall response of the structure if the stress distribution in the constrained rotation members is mainly axial, and so the bending and shearing stresses are a "secondary," rapidly decaying effect near the joints. This will generally be the case for a kinematically determinate structure. The stiffness of this structure is mainly derived from its *axial mode* of action, where its members are either in uniform tension or uniform compression, as the stiffness provided by the *bending mode* of action is several orders of magnitude smaller.

Thus, the global behavior of the structure is fully captured by the pin-jointed model of the structure, whereas its local behavior – which is important to determine local stress concentrations, and hence to check the safety of the structure against fracture and failure – needs a detailed stress analysis of the connections.

Figure 1.1 shows three planar structures used for a static equilibrium experiment by the engineering undergraduates at the University of Cambridge: the geometrical layout of the three structures is the same, but their construction is different. The first structure is made of thin-walled, square section steel tubes with welded connections; the second is made of extruded aluminum-alloy profiles bolted to gusset plates; and the third is made of thin-walled carbon-fiber-reinforced-plastic (CFRP) tubes (made by winding a carbon filament coated with epoxy onto a mandrel and curing the epoxy in a furnace) bonded to aluminum-alloy joint fittings that are connected by steel pins.

During the experiment, the students measure the axial forces in the members of the trusses by comparing the strain in each member of a truss to the strain in the directly-loaded, vertical link under the truss – of identical construction to the rest of the truss.

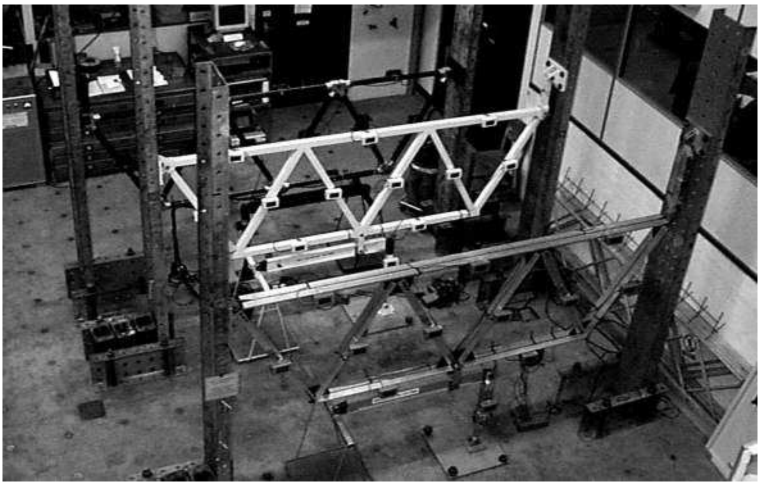

Figure 1.1 Photographs of three truss models in Structures Laboratory at University of Cambridge.

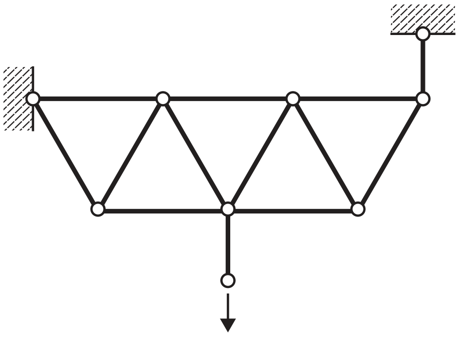

Figure 1.2 Idealized, pin-jointed model of truss structures shown in Fig. 1.1.

They find that the axial forces in corresponding members are practically identical.[1] They also find that the values of the axial forces can be accurately estimated by analyzing the truss shown in Fig. 1.2.

1.2 Rigidity Theory

The key question that we want to deal with is *whether or not a given pin-jointed framework is rigid*.[2] This straightforward question is a surprisingly difficult one to

[1] Note that this experiment is concerned only with the linear-elastic behavior of the three structures. If the loads were increased into the nonlinear range, and up to failure, the behavior of the three structures would no longer be identical.

[2] By rigid we mean a structure that does not deform at all if is assumed that the bars are inextensional, i.e., do not change their length.

answer. Indeed, in many cases it can be answered in full only by carrying out a detailed analysis, or by testing a physical model of the structure.

In two dimensions, i.e., for the case of structures that lie in a plane, the easiest way of constructing a rigid truss is by arranging the bars to form a sequence of triangles. A triangle consisting of pin-jointed bars is the simplest two-dimensional rigid structure, and two triangles with a side in common also form a rigid structure in two dimensions (but not in three dimensions, as one of the triangles can move out of plane by rotating about the common side).

To extend this approach to three dimensions, i.e., to structures in a three-dimensional (Euclidean) space, one can use the simplest three-dimensional structure that is rigid, the tetrahedron (there will be more on this later), or alternatively one can form a closed surface that is completely triangulated. These approaches are often followed in the design of practical structures, but there are also many rigid structures that are not triangulated. Hence, it is of great importance to have a general way of telling whether or not a general three-dimensional structure is rigid. A general method to find the answer computationally will be given in Section 1.5.

A simpler question, that can be answered much more directly, is whether *a pin-jointed structure contains a sufficient number of members to be rigid*. The answer is to count the total number of degrees of freedom of its joints and to subtract the number of degrees of freedom suppressed by applying kinematic constraints to the joints, and by connecting pairs of joints by means of bars.

In two dimensions, each joint has two degrees of freedom, corresponding to two independent translation components, and hence for a structure with j joints the total number of degrees of freedom is $2j$. Denoting by k the total number of kinematic constraints, where, for example, connecting a joint to a foundation counts as two because it suppresses both translation components, and by b the total number of pin-jointed bars – each bar counts one as it imposes a single "distance" constraint between the joints it connects – we require that

$$2j - k - b \leq 0 \tag{1.1}$$

This is known as Maxwell's equation (Maxwell, 1864). Consider, for example, the structure shown in Fig. 1.3(a). It consists of four triangles, the first of which is connected to a foundation, and hence it is obviously a rigid structure. Substituting $j = 6, k = 4, b = 8$ (obviously, there is no need for a bar between the two foundation joints) into Eq. 1.1 we obtain

$$2 \times 6 - 4 - 8 = 0$$

Hence, we conclude that this structure has (just) enough bars to be rigid.

It is important to realize that a structure that has enough bars to be rigid may not in fact be rigid, as its bars may be "incorrectly" placed. For example, if in Fig. 1.3(a) we re-locate the bar bracing the left-hand square, so that the right-hand square is now doubly-braced, as shown in Fig. 1.3(b), we obtain a structure that still satisfies Eq. 1.1 and yet is clearly not rigid. In this case we have a single-degree-of-freedom mechanism, Fig. 1.4(a). A structure that admits no mechanisms is called *kinematically determinate*.

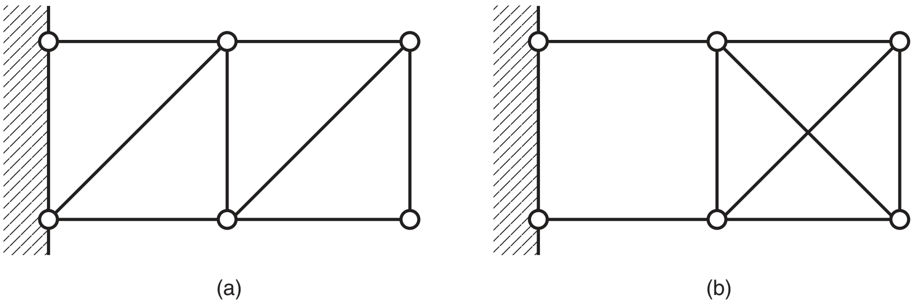

Figure 1.3 Examples of two-dimensional pin-jointed structures that are (a) fully triangulated and hence rigid, (b) a mechanism.

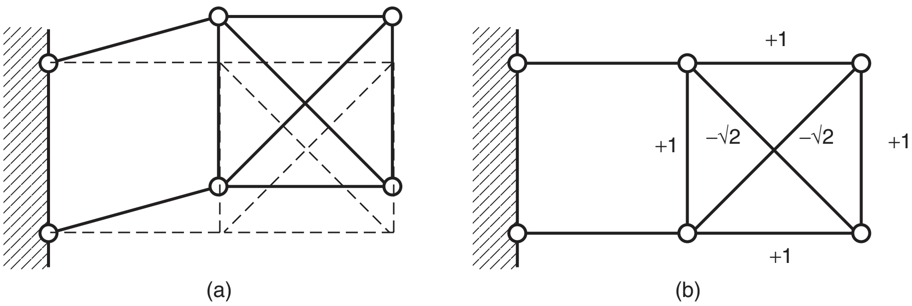

Figure 1.4 Mechanism (exaggerated amplitude of a small-amplitude motion) and state of self-stress of structure shown in Fig. 1.3(b).

Note that the doubly braced square on the right-hand side of the structure in Fig. 1.3(b) admits a state of self-stress, i.e., there is a set of non-zero bar forces that are in equilibrium with zero external forces, as shown in Fig. 1.4(b). A structure that admits no states of self-stress is called *statically determinate*.

Denoting by m the number of independent mechanisms of a structure, and by s the number of states of independent states of self-stress, for the structure of Fig. 1.3(a) we have $s = 0$ and $m = 0$ (statically and kinematically determinate), whereas for the structure of Fig. 1.3(b) we have $s = 1$ and $m = 1$ (statically and kinematically indeterminate). Here, by *independent* we mean that if any mechanism is represented by a vector, whose components correspond to the tangent motions of the joint, and any state of self-stress by a vector whose components correspond to the bar forces, it is not possible to obtain one of the vectors as a linear combination of the others.

So, Maxwell's equation in the form of Eq. 1.1 is only a *necessary condition* for the kinematic determinacy of pin-jointed structures, but not a *sufficient condition*. It will be shown in Section 1.5 that the general, and most useful way, of writing Maxwell's equation is:

$$dj - b - k = m - s \tag{1.2}$$

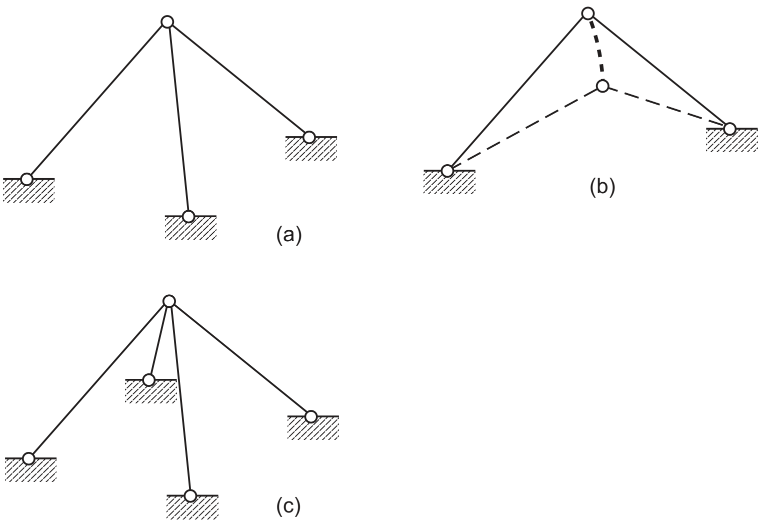

Figure 1.5 Examples of simple three-dimensional trusses (Pellegrino and Calladine, 1986).

where $d = 2$, or 3 depending on the dimensions of the (Euclidean) space in which the structure is considered.

Consider the three-dimensional structures, $d = 3$, shown in Fig. 1.5. The tripod structure in Fig. 1.5(a) has a single free joint plus three fully constrained joints; so $j = 4$ and $k = 9$. The unconstrained joint is connected by three non-coplanar bars, $b = 3$, to the foundation joints. It has no states of self-stress, $s = 0$, as the condition for the joint to be in equilibrium in three different directions without external forces requires that the bar forces be zero. Substituting into Eq. 1.2 gives:

$$3 \times 4 - 3 - 9 = 0 = m - 0 \tag{1.3}$$

from which the number of mechanisms is $m = 0$.

Having established that $s = 0$ for the structure of Fig. 1.5(a), obviously s will remain unchanged if a bar is removed, Fig. 1.5(b). Hence, for this structure $j = 3$, $k = 6$, and $b = 2$. Substituting into Maxwell's equation:

$$3 \times 3 - 2 - 6 = m - 0 \tag{1.4}$$

which gives $m = 1$. The mechanism involves a rotation of the two bars about an axis passing through the two foundation joints, as shown in Fig. 1.5(b).

By an analogous argument, the structure of Fig. 1.5(c), which is obtained by adding a bar to the structure of Fig. 1.5(a), has $m = 0$ and, from Maxwell's equation, $s = 1$.

Figure 1.6 shows two examples of pin-jointed structures that are topologically identical to the structure in Fig. 1.5(a), i.e., they have the same numbers of joints, bars, and constraints; *but now the bars are coplanar*. These structures admit a state of self-stress, e.g., a tension in the two inclined members equilibrated by a compression in the vertical member.

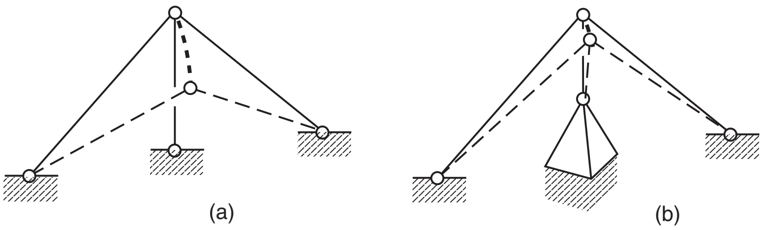

Figure 1.6 Examples of pin-jointed structures that are both statically and kinematically indeterminate. In (a) the three bars are coplanar *and* the three foundation joints are collinear (Pellegrino and Calladine, 1986).

Since the left-hand side of Eq. 1.3 is unchanged, but now $s = 1$, here $m = 1$. In both structures the mechanism is identical to that shown in Fig. 1.5(b), but whereas Fig. 1.6(a) is a *finite mechanism*, in Fig. 1.6(b) only a small-amplitude motion of the mechanism is possible. This is because the central foundation joint is aligned with the other two in Fig. 1.6(a) but not in Fig. 1.6(b).

The truss structure in Fig. 1.6(b) is a simple example of an *infinitesimal mechanism*. If a structure of this kind is made with infinitely rigid members and perfectly fitting joints, it would admit only an infinitesimal motion of its mechanism. In practice, of course, its members will be elastic and there will be some tolerance in the joints; hence, the stiffness of the structure will be of a "lower order" than that of a normal, kinematically determinate structure.

Note that in the mathematics literature on structural rigidity a rigid structure is any structure that is either kinematically determinate or indeterminate but with mechanisms that are only infinitesimal (Connelly, 1993). Engineers tend to use the definition of rigid structures adopted here, which includes the smaller class that admit no mechanisms at all.

The existence of structures with infinitesimal mechanisms was first discovered by J. Clerk Maxwell (1864), but it was only more recently that it was realized that they can be given a first-order (geometric) stiffness through a state of prestress (Calladine, 1986). This property has been successfully exploited in the design of prestressed cable nets, see Section 3.2, and tensegrity structures, see Section 4.3.

1.2.1 Polyhedral Trusses

Figure 1.7 shows five trusses based on the five platonic polyhedra, more details of which can be found in Appendix A.2.

The simplest of these structures is the tetrahedral truss; from Table A.1 $j = 4, b = 6$, and $k = 0$. Hence, Maxwell's equation gives:

$$3 \times 4 - 6 - 0 = 6 = m - s \tag{1.5}$$

Because there are only three noncoplanar bars meeting at each joint, for which three equations of equilibrium can be written, the bar forces have to be equal zero if the

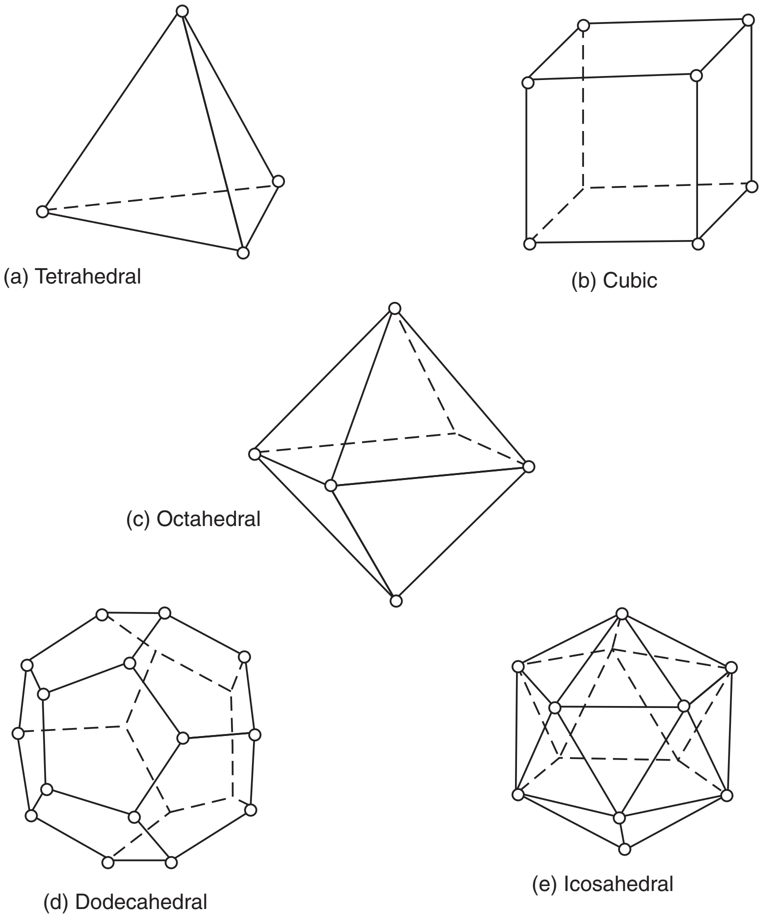

Figure 1.7 Regular polyhedral trusses.

external loads are zero. Therefore $s = 0$ and so, from Eq. 1.5, $m = 6$. Because the truss has six rigid-body mechanisms as a free body in three-dimensional space, i.e., three independent translations and three rotations, these are the only mechanisms of the truss. Denoting by m' the number of independent *internal mechanisms*, we have $m' = 0$ for the tetrahedral truss, i.e., it is *internally rigid*.

Next, consider the cubic truss, Fig. 1.7(b). From Table A.1, $j = 8, b = 12$, and $k = 0$; Maxwell's equation gives:

$$3 \times 8 - 12 - 0 = 12 = m - s \tag{1.6}$$

Because $s = 0$, which can be shown by the same argument as for the tetrahedral truss, Eq. 1.6 gives:

$$m = 12$$

Table 1.1 Static and (internal) kinematic determinacy of polyhedral trusses.

Shape	s	m'
Tetrahedron	0	0
Cube	0	6
Octahedron	0	0
Dodecahedron	0	24
Icosahedron	0	0

of which six are rigid-body motions, as above, and the remaining six are internal mechanisms. For example, six independent mechanisms are obtained by deforming each square of the truss into a rhombus.

Repeating the same analysis for the remaining trusses it is found that $m - s = 6$ for the octahedral and icosahedral trusses, but $m - s = 30$ for the dodecahedral truss. Then, since it can be shown that $s = 0$ for all of them – although the proof is not straightforward for the octahedral and icosahedral trusses – it can be concluded that the octahedral and icosahedral trusses are internally rigid, but not the dodecahedral truss. These results are summarized in Table 1.1.

Note that the five trusses based on the platonic polyhedra can all be regarded as tessellations of triangles, squares and pentagons on a sphere. Also note that only the tessellations of triangles have turned out to be rigid; the cube and the dodecahedron – consisting of tessellations of squares and pentagons, respectively – have many mechanisms. This result was to be expected, in light of the earlier comment, in Section 1.1, on the rigidity of triangulated surfaces.

1.2.2 Cauchy's Theorem

The rigidity of a truss consisting of a tessellation of triangles that lie on a sphere follows from a theorem proved by Cauchy, together with several other theorems for polygons and polyhedra. Theorem 13 of Cauchy (1813) states that:

In a convex polyhedron with invariable faces the angles at the edges are also invariable, so that with the same faces one can build only a polyhedron symmetrical to the first one.

Thus, every convex polyhedron with rigid faces will be rigid and, since the simplest way of forming a rigid face with pin-jointed bars is to use a triangle, Cauchy's theorem can also be stated in the specialized form:

Every convex polyhedral surface is rigid if all of its faces are triangles.

An example of a truss whose rigidity follows from Cauchy's theorem is shown in Fig. 1.8. This structure has been obtained by considering arcs of great circles that join the vertices of an icosahedron – which by definition lie on a sphere – and by locating an additional joint at the mid-point of each arc. Then, each joint has been connected

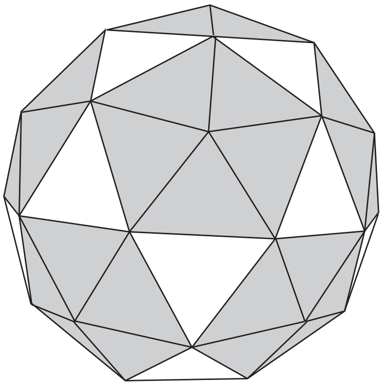

Figure 1.8 Truss structure obtained by adding a series of mid-arc nodes to an icosahedron. The nodes of the original icosahedron are at the center of the pentagons.

with a bar to all of its neighbors. The resulting truss structure has $j = 42$, as 12 joints coincide with the vertices of the icosahedron, plus there are 30 joints at the mid-points of the great circle arcs. The number of bars is equal to twice the number of edges of the icosahedron, E, plus three times the number of faces, F, whose values are given in Table A.1. Hence $b = 2E + 3F = 120$. Maxwell's equation gives $m - s = 6$ and, since $s = 0$, the only mechanisms are the six rigid motions.

Despite the restriction in Cauchy's theorem, that the surface should be convex, mathematicians had conjectured for over 150 years that in fact all surfaces consisting of triangles are rigid, even those surfaces that are not convex[3]. This was known as the "rigidity conjecture," which was finally proven to be wrong by a counter-example devised by Connelly (1978).

Since it took so long to find a counter-example, we can safely state that "almost all" triangulated surfaces are rigid. This means that one is very unlikely to ever encounter a simply connected triangulated surface of any shape that is not rigid.

1.2.3 Flexible "Sphere"

Several examples of concave triangulated structures that admit an infinitesimal motion were found over the years, but none whose motion was finite. Connelly's discovery of a counter-example to the "rigidity conjecture," which he called a *flexible sphere*, led to the subsequent discovery of several such structures by other authors. One of these examples is shown next.

Figure 1.9 shows a model, made from the cutting pattern in Fig. 1.10: the pattern is meant to be scaled up on a photocopying machine so that the edge numbers should be lengths in centimeters. This gives a size that is easy to work with. On the pattern, curved

[3] Note that the surfaces that we are considering here are *simply connected*, i.e., topologically identical to a sphere. Toroidal surfaces, for example, are excluded.

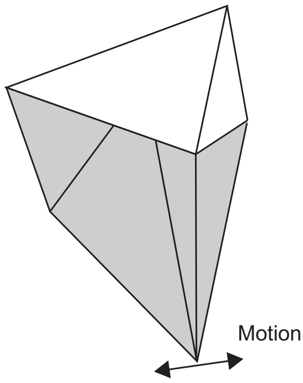

Figure 1.9 Perspective view of Connelly–Steffen "flexible sphere" made from cutting pattern in Fig. 1.10 (Dewdney, 1991).

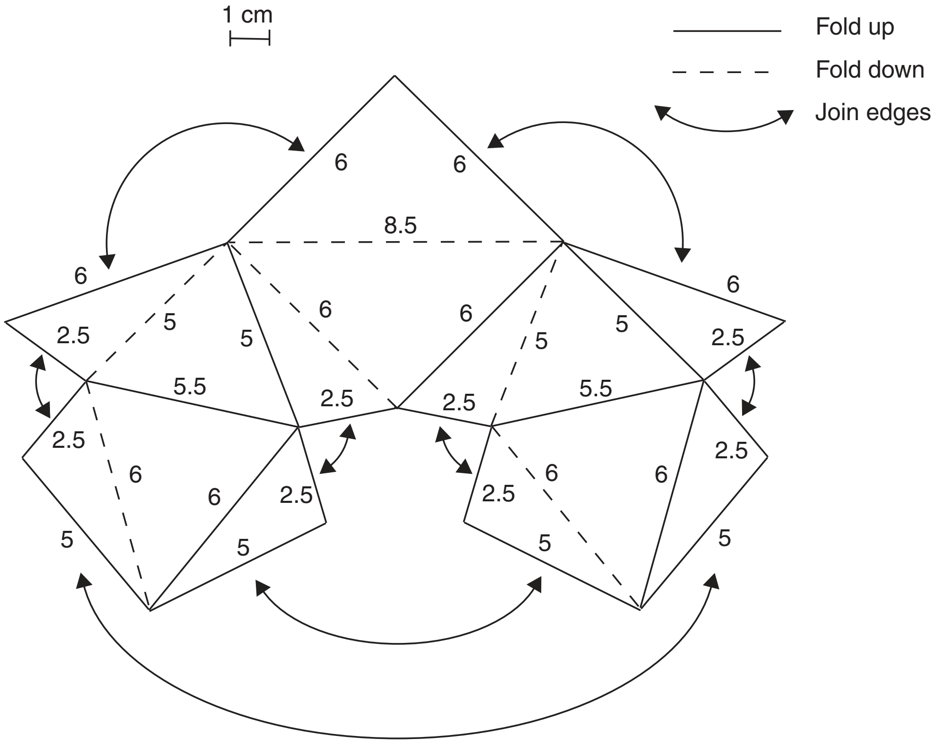

Figure 1.10 Cutting pattern for Connelly–Steffen "flexible sphere" (Dewdney, 1991).

arrows indicate pairs of edges that should be attached, e.g., by leaving a tab on one side and gluing it under the other side.

After making your own model, try rotating the upper triangle relative to the lower one (not shown in Fig. 1.9): it will move without any resistance until two internal triangles come into contact. Note that while the structure moves, there is no sign of it stiffening