ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

EDITED BY G.-C. ROTA

Volume 57

Skew fields

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Skew fields

Theory of general division rings

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To my grandson

James Abraham Aaronson

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PREFACE

When *Skew Field Constructions* appeared in 1977 in the London Mathematical Society Lecture Note Series, it was very much intended as a provisional text, to be replaced by a more definitive version. In the intervening years there have been some new developments, but most of the progress has been made in the simplification of the proofs of the main results. This has made it possible to include complete proofs in the present version, rather than to have to refer to the author's *Free Rings and their Relations*. An attempt has also been made to be more comprehensive, but we are without a doubt only at the beginning of the theory of skew fields, and one would hope that this book will offer help and encouragement to the prospective builders of such a theory. The genesis of the theory was described in the original preface (see the extract following this preface); below we briefly outline the subjects covered in the present book.

The first four chapters are to a large extent independent of each other and can be read in any order, referring back as necessary. Ch. 1 gives the general definitions and treats the Ore case as well as various necessary conditions for the embedding of rings in skew fields. From results in universal algebra it follows that necessary and sufficient conditions for such an embedding take the form of quasi-identities. Later, in Ch.4, we shall find the explicit form of these quasi-identities, and in Ch. 6 we shall see that this set must be infinite. The rest of Ch. 1 gives the definition and basic properties of free algebras and free ideal rings, which play a major role later. It also includes some technical results on the association of matrices and it introduces an important technical tool: the matrix reduction functor.

Ch. 2 studies skew polynomial rings and the fields formed from them, as well as power series rings and generalizations such as the Malcev–Neumann construction, and the author's results on fields of fractions for a class of filtered rings. Ch. 3 is devoted to the Galois theory of skew fields, now almost classical, with applications to (left or) right polynomial equations over skew fields, and special cases of extensions, such as pseudo-linear extensions and cyclic Galois extensions.

Ch. 4 is in many ways the central chapter. The process of forming fields of fractions or more generally epic R-fields for a ring R is described in terms of the *singular kernel*, i.e. the set of matrices that become singular over the field. It is shown how any epic R-field can be constructed from its singular kernel, while the latter has a simple description as prime matrix ideal. This leads to explicit conditions for the existence of a field of fractions. In particular, the rings with a fully inverting homomorphism to a field are characterized as Sylvester domains and it is shown that every semifir has a universal field of fractions. Of the earlier sections only 1.6 is needed here.

Ch. 5 describes the coproduct construction and the results proved here are basic for much that follows. It is also the most technical chapter and the reader may wish to postpone the details of the proofs in 5.1–3 to a second reading, but he should familiarize himself with the results. They are applied in the rest of the chapter to give the HNN-construction for fields and for rings, to study the effect of adjoining generators and relations, particularly matrix relations, and to construct field extensions with different left and right degrees (Artin's problem). Ch. 6 deals with some general questions. There is a study of free fields; here the specialization lemma is an essential tool. Other topics include the word problem and existentially closed fields.

Ch. 7 on rational identities is mainly devoted to Bergman's theory of specializations between rational meets of X-fields; it is independent of most of the rest and can be read at any stage.

In Ch. 8 the rather fragmentary state of knowledge of singularities (which in the general theory take the place of equations in the commutative theory) is surveyed, with an account of the problems to be overcome to launch a form of non-commutative algebraic geometry. Ch. 9 deals with valuations and orderings on skew fields from the point of view of the general construction of Ch. 4 and it shows for example how to construct valuations and orderings on the free field.

The exercises are intended for practice but serve also to present additional developments in brief form, as well as some open problems. Some historical background is given in the Notes and comments.

The theory of division algebras (finite-dimensional over a field) is very much further advanced than the general theory of skew fields, and a comprehensive account including a full treatment of division algebras would have thrown the whole out of balance and resulted in a very bulky tome. For this reason that topic has largely been left aside; this was all the more reasonable as the subject matter is much more accessible, and no doubt will be even more so with the forthcoming publication of the treatise by Jacobson and Saltman.

Nearly all the material in this volume has been presented to the Ring Theory Study Group at University College London and I am grateful to the members of this group for their patience and help. I would like to thank Mark L. Roberts for his comments on early chapters and George M. Bergman for his criticism of *Skew Field Constructions*, which has proved most useful. My thanks also go to the staff of the Cambridge University Press for their help in transforming the manuscript into a book, with a particular word of thanks to their copy editor Mr Peter Jackson, who corrected not merely grammatical but also mathematical slips. As always, I shall be glad to receive any constructive criticism from readers, best of all, news of progress on the many open problems.

London, February 1995 P. M. Cohn

From the preface to *Skew Field Constructions*

The history of skew fields begins with quaternions, whose discovery (in 1843) W. R. Hamilton regarded as the climax of his far from ordinary career. But for a coherent theory one has to wait for the development of linear associative algebras; in fact it was not until the 1930's that a really comprehensive treatment of skew fields (by Hasse, Brauer, E. Noether and Albert) appeared. It is an essential limitation of this theory that only skew fields finite-dimensional over their centres are considered.

Although general skew fields have made an occasional appearance in the literature, especially in connexion with the foundations of geometry, very little of their properties was known until recently, and even particular examples were not easy to come by. The first well known case is the field of skew power series used by Hilbert in 1899 to illustrate the fact that a non-archimedean ordered field need not be commutative. There are isolated papers in the 1930's, 1940's and 1960's (Moufang, Malcev, B. H. Neumann, Amitsur and the author) showing that the free algebra can be embedded in a skew field, but the development of the subject is hampered by the fact that one has no operation that can be performed on skew fields (over a given ground field) and again produces a skew field containing the original ones. In the commutative case one has the tensor product, which leads to a ring from which fields can then be obtained as homomorphic images. The corresponding object in the general case is the free product and in the late 50's the author tried to prove that this could be embedded in a skew field. This led to the development of firs (= free ideal rings); it could be shown (1963) that any free product of skew fields is a fir, but it was not until 1971 that the original aim was achieved, by proving that every fir is embeddable in a skew field, and in fact has a universal field of fractions. Combining these results, one finds that any free product or 'coproduct' of fields has a universal field of fractions, or a *field coproduct*, as we shall call it. It is this result which forms the starting point for these lectures.

As the name indicates, these really are lecture notes, though not for a single set of lectures. For this reason they may lack the polish of a book, but it is hoped that they have not entirely lost the directness of a lecture. The material comes from courses I have given in Manchester and London; some parts follow rather closely lectures given at Tulane University (1971), the University of Alberta (1972), Carleton University (1973), Tübingen (1974), Mons (1974), Haifa Technion (1975), Utrecht (1975) and Ghent (1976). It is a pleasure to acknowledge the hospitality of these institutions, and the stimulating effect of such critical audiences.

NOTE TO THE READER

The reader of this book is expected to have a fair background in algebra, particularly ring theory and commutative field theory. Any standard results needed are usually quoted from the author's *Algebra* (referred to as A.1, 2, 3, see the Bibliography).

All theorems, propositions, lemmas and corollaries are numbered consecutively in a single series in each section; thus Th. 2.2 is followed by Prop. 2.3 in Section 1.2, and outside Ch. 1 they are referred to as Th. 1.2.2, Prop. 1.2.3 respectively. Occasional results needed but not proved are usually given letters, e.g. Th. 6.A. The end (or absence) of a proof is indicated by ∎. Most sections have exercises; open-ended (or open) problems are marked °. Unexplained notations can be found in the list of standard notations on p. 473.

References to the bibliography are by the author's name and the last two digits of the year of publication if after 1900, e.g. Ore [31], Cramer [1750], with primes to distinguish publications by the same author in the same year.