1. Unconditional and Absolute Summability in Banach Spaces

The Dvoretzky-Rogers Theorem

Recall that a sequence (x_n) in a normed space is *absolutely summable* if $\sum_n \|x_n\| < \infty$, and is *unconditionally summable* if $\sum_n x_{\sigma(n)}$ converges, regardless of the permutation σ of the indices. It is traditional to say that the series $\sum_n x_n$ is absolutely (unconditionally) convergent if the sequence (x_n) is absolutely (unconditionally) summable.

A theorem of Dirichlet from elementary analysis asserts that a scalar sequence is absolutely summable precisely when it is unconditionally summable. Simple natural adjustments to the proof show that this theorem extends to the setting of any finite dimensional normed space.

What happens in infinite dimensional spaces? Without completeness we can get nowhere.

1.1 Proposition: *A normed space is a Banach space if and only if every absolutely summable sequence is unconditionally summable.*

This elementary old standby finds frequent use in proofs of completeness, and a brief indication of its proof is worthy of our attention.

Proof. To show completeness we need to prove that every Cauchy sequence (x_n) is convergent. For this it suffices to find a convergent subsequence, a task which is not difficult since any 'sufficiently rapid' Cauchy subsequence will do the trick. For example, choose an increasing sequence of positive integers (n_k) so that if $y_k = x_{n_{k+1}} - x_{n_k}$, then $\|y_k\| \leq 2^{-k}$. As (y_k) is absolutely summable, it is (unconditionally) summable. The convergence of (x_{n_k}) now follows from the identity $x_{n_1} + y_1 + \ldots + y_k = x_{n_{k+1}}$.

Conversely, it is plain that in a Banach space the sequence of partial sums of an absolutely summable sequence is a Cauchy sequence, and so convergent. An application of Dirichlet's Theorem to the sum of the norms allows us to obtain *unconditional* convergence. QED

In standard infinite dimensional spaces there are usually easy examples of unconditionally summable sequences which fail to be absolutely summable. For instance, in ℓ_2 the sequence $(\lambda_n e_n)$ – where e_n is the n-th unit coordinate vector – is unconditionally summable when $(\lambda_n) \in \ell_2$, but not absolutely summable unless $(\lambda_n) \in \ell_1$.

Remarkably enough, this example can be transplanted, after suitable modification, to any infinite dimensional Banach space. The operation requires a modicum of solid effort.

1.2 Dvoretzky-Rogers Theorem: *Let X be an infinite dimensional Banach space. Then no matter how we choose $(\lambda_n) \in \ell_2$ there is always an unconditionally summable sequence (x_n) in X with $\|x_n\| = |\lambda_n|$ for all n.*

So, choosing (λ_n) in ℓ_2 but not in ℓ_1, we have sequences in infinite dimensional Banach spaces which are unconditionally summable but not absolutely summable.

The heart of the Dvoretzky-Rogers Theorem is a striking geometrical lemma.

1.3 Lemma: *Let E be a $2n$-dimensional Banach space. There exist n vectors $x_1,\dots,x_n \in B_E$, each of norm $\geq 1/2$, such that regardless of the scalars $\lambda_1,\dots,\lambda_n$ we have*

$$\Big\|\sum_{j\leq n}\lambda_j x_j\Big\| \leq \Big(\sum_{j\leq n}|\lambda_j|^2\Big)^{1/2}.$$

Proof. First we fix some (standard) notations. If w is a linear map between two k-dimensional vector spaces, each with a chosen basis, then $\det(w)$ and $\operatorname{tr}(w)$ denote the determinant and trace of the matrix representing w with respect to these chosen bases. Changing the bases will change $\det(w)$ by a constant factor but will leave $\operatorname{tr}(w)$ unchanged.

Our objective is to find a norm one isomorphism $u : \ell_2^{2n} \to E$ satisfying

$$(*) \qquad |\operatorname{tr}(u^{-1}v)| \leq 2n\cdot\|v\|$$

for all operators $v : \ell_2^{2n} \to E$.

Locating u is no problem: take any u for which

$$\det(u) = \max\big\{|\det(v)| : v \in \mathcal{L}(\ell_2^{2n}, E),\ \|v\| = 1\big\}.$$

The compactness of the unit sphere of $\mathcal{L}(\ell_2^{2n}, E)$ and the continuity of $\det(\cdot)$ guarantee the existence of such an operator u.

Establishing $(*)$ takes more care; we use a perturbation argument. Let ε be a non-zero scalar and $v \in \mathcal{L}(\ell_2^{2n}, E)$. Then, by the choice of u and the nature of determinants,

$$\frac{|\det(u+\varepsilon v)|}{\|u+\varepsilon v\|^{2n}} \leq \det(u),$$

so that

$$|\det(u+\varepsilon v)| \leq \det(u)\cdot\|u+\varepsilon v\|^{2n} \leq \det(u)\cdot(1+|\varepsilon|\cdot\|v\|)^{2n}.$$

Let id denote the identity on ℓ_2^{2n}. The invertibility of u gives

$$\begin{aligned}|\det(u+\varepsilon v)| &= \det(u)\cdot|\det(id+\varepsilon u^{-1}v)|\\ &= \det(u)\cdot|1+\varepsilon\cdot\operatorname{tr}(u^{-1}v)+c(\varepsilon)|\end{aligned}$$

where $|c(\varepsilon)| = O(|\varepsilon|^2)$ as $\varepsilon \to 0$. In tandem, these observations tell us that

$$|1+\varepsilon\cdot\mathrm{tr}\,(u^{-1}v)+c(\varepsilon)| \le (1+|\varepsilon|\cdot\|v\|)^{2n} = 1+2n\cdot|\varepsilon|\cdot\|v\|+O(|\varepsilon|^2)$$

for small ε. Choosing ε judiciously enough to satisfy $\varepsilon\cdot\mathrm{tr}\,(u^{-1}v)=|\varepsilon\cdot\mathrm{tr}\,(u^{-1}v)|$, we soon have for small $|\varepsilon|$

$$1+|\varepsilon|\cdot|\mathrm{tr}\,(u^{-1}v)| \le 1+|\varepsilon|\cdot 2n\cdot\|v\|+O(|\varepsilon|^2),$$

from which

$$|\mathrm{tr}\,(u^{-1}v)| \le 2n\cdot\|v\|+O(|\varepsilon|).$$

Passing with ε to zero we obtain $(*)$.

Now if P is any orthogonal projection on ℓ_2^{2n} with m-dimensional range, we can exploit $(*)$ to get

$$m = \mathrm{tr}\,(P) = \mathrm{tr}\,(u^{-1}uP) \le 2n\cdot\|uP\|.$$

In other words, $\|uP\|\ge m/2n$.

At last we can address the problem of selecting suitable $x_1,\ldots,x_n$ in E. The key is to choose appropriate orthonormal vectors $y_1,\ldots,y_n$ in ℓ_2^{2n} and then to set $x_j=u(y_j)$ for each j.

Since $\|u\|=1$, there is a $y_1\in\ell_2^{2n}$ with $\|y_1\|=1$ and $\|uy_1\|=1$. Let P_1 be the orthogonal projection of ℓ_2^{2n} onto the orthogonal complement $[y_1]^\perp$ of (the span of) y_1. Then $\|uP_1\|\ge(2n-1)/2n$, so there is a $y_2\in[y_1]^\perp$ with $\|y_2\|=1$ and $\|uy_2\|=\|uP_1y_2\|\ge(2n-1)/2n$. Let P_2 be the orthogonal projection of ℓ_2^{2n} onto the orthogonal complement $[y_1,y_2]^\perp$ of y_1 and y_2. Then $\|uP_2\|\ge(2n-2)/2n$, so there is a $y_3\in[y_1,y_2]^\perp$ with $\|y_3\|=1$ and $\|uy_3\|=\|uP_2y_3\|\ge(2n-2)/2n$. — Continue.

After n steps we have orthonormal vectors $y_1,\ldots,y_n$ in ℓ_2^{2n}. Set $x_j=uy_j$ for $1\le j\le n$. Then $\|x_j\|\ge(2n-j+1)/2n\ge 1/2$ for all j, and at the same time, if $\lambda_1,\ldots,\lambda_n$ are scalars then

$$\Big\|\sum_{j\le n}\lambda_jx_j\Big\|=\Big\|u\big(\sum_{j\le n}\lambda_jy_j\big)\Big\|\le\|u\|\cdot\Big\|\sum_{j\le n}\lambda_jy_j\Big\|=\big(\sum_{j\le n}|\lambda_j|^2\big)^{1/2},$$

by the orthonormality of the y_j's. QED

This lemma will recur in several important improvements; see 14.2 and 19.15.

To prove the Dvoretzky-Rogers Theorem, it is convenient to use an alternative formulation of unconditional summability.

1.4 Lemma: *A sequence (x_n) in a Banach space is unconditionally summable if and only if it is sign summable, that is $\sum_n\varepsilon_nx_n$ converges for all choices of signs $\varepsilon_n=\pm1$.*

The proof of this lemma requires some rather formal manipulations, which are best seen in the context of other equivalent variations of unconditional summability. We shall confront these shortly (1.5 through 1.9) but for now we take the lemma on trust.

Proof of the Dvoretzky-Rogers Theorem. Fix $(\lambda_n) \in \ell_2$ and choose positive integers $n_1 < n_2 < \dots$ such that, for each $k \in \mathbf{N}$,

$$\sum_{n \geq n_k} |\lambda_n|^2 \leq 2^{-2k}.$$

Since X is infinite dimensional, the geometrical lemma 1.3 applies with impunity to all dimensions. We can therefore find a sequence of vectors (y_n) in B_X, each of norm $\geq 1/2$, such that for any scalar sequence (α_n) and any k we have

$$\Big\| \sum_{n=n_k}^{N} \alpha_n y_n \Big\| \leq \Big(\sum_{n=n_k}^{N} |\alpha_n|^2 \Big)^{1/2},$$

no matter how we select $n_k \leq N < n_{k+1}$. We weight the y_j's by setting $x_j = \lambda_j y_j / \|y_j\|$. Take note: regardless of the signs $\varepsilon_n = \pm 1$ and regardless of $n_k \leq N < n_{k+1}$ we have

$$\Big\| \sum_{n=n_k}^{N} \varepsilon_n x_n \Big\| \leq \Big(\sum_{n=n_k}^{N} \frac{|\lambda_n|^2}{\|y_n\|^2} \Big)^{1/2} \leq 2^{-k+1}.$$

It follows that the partial sums of $(\varepsilon_n x_n)$ are Cauchy. Hence (x_n) is sign summable, and so unconditionally summable. Of course, our weighty scaling of the y_n's ensures that $\|x_n\| = |\lambda_n|$ for all n. QED

Unconditional Convergence and the Orlicz-Pettis Theorem

We have now seen that in infinite dimensional Banach spaces the notions of unconditional and absolute summability are never equivalent. Just what *can* we say about unconditional summability?

In the next few pages we amass an extensive arsenal of useful equivalences to unconditional summability. Though some of these are elementary in nature others, like the Orlicz-Pettis Theorem 1.8, are subtle and surprising results.

To start, we shall rely only on manipulative skill, and we shall confine ourselves to careful reworkings of the scalar-valued case.

1.5 Theorem: *For a sequence (x_n) in a Banach space, the following are equivalent:*

(i) *(x_n) is unconditionally summable.*

(ii) *(x_n) is unordered summable, that is, for any $\varepsilon > 0$ there is a positive n_ε such that whenever M is a finite subset of $\mathbf{N}$ with $\min M > n_\varepsilon$ we have $\| \sum_{n \in M} x_n \| < \varepsilon$.*

(iii) *(x_n) is subseries summable, that is, for any strictly increasing sequence (k_n) of positive integers, $\sum_n x_{k_n}$ is convergent.*

(iv) *(x_n) is sign summable.*

Proof. (i)⇒(ii): Arguing contrapositively, we assume that (ii) is false and look for a permutation σ of $\mathbf{N}$ which makes the partial sums of $(x_{\sigma(n)})$ fail to be Cauchy. There is an exceptional $\delta > 0$ such that, regardless of $m \in \mathbf{N}$, there is always a finite set $M \subset \mathbf{N}$ with $\min M > m$ and $\|\sum_{k\in M} x_k\| \geq \delta$. This allows us to construct a sequence (M_n) of finite subsets of $\mathbf{N}$ such that, for all $n \in \mathbf{N}$,

$$\max M_n < \min M_{n+1} \qquad \text{and} \qquad \Big\|\sum_{k\in M_n} x_k\Big\| \geq \delta.$$

But then, if σ is a permutation of $\mathbf{N}$ which maps each (integer) interval $[\min M_n, \min M_n + |M_n|)$ to M_n, the partial sums of $(x_{\sigma(n)})$ cannot be Cauchy.
(ii)⇒(i): Let σ be a permutation of $\mathbf{N}$. Fix $\varepsilon > 0$ and choose $n_\varepsilon \in \mathbf{N}$ as in the definition of unordered summability. Certainly there is an $m_\varepsilon \in \mathbf{N}$ such that $\{1,\dots,n_\varepsilon\} \subset \sigma(\{1,\dots,m_\varepsilon\})$, and this tells us that if $q > p > m_\varepsilon$, then $\|\sum_{n=p}^q x_{\sigma(n)}\| < \varepsilon$. In other words, $(x_{\sigma(n)})$ is summable.
(ii)⇒(iii): Fix $\varepsilon > 0$ and choose $m_\varepsilon \in \mathbf{N}$ so that $\|\sum_{n\in M} x_n\| < \varepsilon$ for every finite subset M of $\mathbf{N}$ with $\min M > m_\varepsilon$. Now if (k_n) is a strictly increasing sequence of natural numbers we have $k_n \geq n$ for each n, so that if $q > p > m_\varepsilon$, then $\|\sum_{n=p}^q x_{k_n}\| < \varepsilon$. In other words, (x_{k_n}) is summable.
(iii)⇒(iv): Let (ε_n) be a sequence of ± 1's. If (x_n) is subseries summable and we set $S^+ = \{n \in \mathbf{N} : \varepsilon_n = 1\}$ and $S^- = \{n \in \mathbf{N} : \varepsilon_n = -1\}$, then it must be the case that both series $\sum_{n\in S^\pm} x_n$ are convergent. Fix $\varepsilon > 0$. Notice that if $p < q$ are natural numbers and $M^\pm = \{n \in S^\pm : p \leq n \leq q\}$, then

$$\sum_{n=p}^q \varepsilon_n x_n = \sum_{n\in M^+} x_n - \sum_{n\in M^-} x_n.$$

It follows from the convergence of $\sum_{n\in S^\pm} x_n$ that $\|\sum_{n\in M^\pm} x_n\| < \varepsilon/2$ for p sufficiently large. Then $\|\sum_{n=p}^q \varepsilon_n x_n\| < \varepsilon$ for such p, and this is enough to ensure sign summability.
(iv)⇒(ii): For this final step, we will again take the contrapositive route. Assuming that (x_n) is not unordered summable, we have no option but to admit the existence of an exceptional $\delta > 0$ and a sequence (M_k) of finite subsets of $\mathbf{N}$ for which $\max M_k < \min M_{k+1}$ and $\|\sum_{n\in M_k} x_n\| \geq \delta$. Assign ε_n the value $+1$ if $n \in \bigcup_k M_k$, and -1 otherwise. The partial sums of $\big((1+\varepsilon_n)x_n\big)$ then fail to be Cauchy, so at least one of the series $\sum x_n$ and $\sum \varepsilon_n x_n$ will find it impossible to converge. QED

We have just seen that unconditional summability of (x_n) yields summability of $(b_n x_n)$ for certain very special sequences in ℓ_∞ – the sequences of signs. These special (b_n)'s are, at least in the real case, just the extreme points of B_{ℓ_∞}, so, in view of the Krein - Milman Theorem, the next result should not come as too much of a surprise.

1.6 Bounded Multiplier Test: *In any Banach space, the sequence (x_n) is unconditionally summable if and only if the sequence $(b_n x_n)$ is summable for every $(b_n) \in \ell_\infty$.*

Proof. One direction is trivial since we know sign summability implies unconditional summability. Accordingly, we fix an unconditionally summable sequence (x_n) in the Banach space X and show that $(b_n x_n)$ is summable when $b = (b_n) \in \ell_\infty$. The completeness of X means that we need only show that $\| \sum_{k=m}^n b_k x_k \|$ converges to zero as $m, n \to \infty$. Using duality,

$$\Big\| \sum_{k=m}^n b_k x_k \Big\| = \sup_{x^* \in B_{X^*}} \Big| \Big\langle x^*, \sum_{k=m}^n b_k x_k \Big\rangle \Big| \le \|b\|_\infty \cdot \sup_{x^* \in B_{X^*}} \sum_{k=m}^n |\langle x^*, x_k \rangle|,$$

and this leads us to try to show that

$$(*) \qquad \lim_{m\to\infty} \sup_{x^* \in B_{X^*}} \sum_{k \ge m} |\langle x^*, x_k \rangle| = 0.$$

Unordered summability is the appropriate medium. If $\varepsilon > 0$ is given, there is an $m_\varepsilon \in \mathbf{N}$ such that for any finite set $M \subset \mathbf{N}$ with $\min M > m_\varepsilon$ we have $\| \sum_{k \in M} x_k \| < \varepsilon$. Let us concentrate on $\mathrm{Re}\,\langle x^*, x_k \rangle$ for a fixed $x^* \in B_{X^*}$. Choose $n > m > m_\varepsilon$ and set

$$M^+ = \{ m \le k \le n : \mathrm{Re}\,\langle x^*, x_k \rangle \ge 0 \}$$

and

$$M^- = \{ m \le k \le n : \mathrm{Re}\,\langle x^*, x_k \rangle < 0 \}.$$

Then

$$\begin{aligned} \sum_{k=m}^n |\mathrm{Re}\,\langle x^*, x_k \rangle| &= \Big| \mathrm{Re}\,\Big\langle x^*, \sum_{k \in M^+} x_k \Big\rangle \Big| + \Big| \mathrm{Re}\,\Big\langle x^*, \sum_{k \in M^-} x_k \Big\rangle \Big| \\ &\le \Big\| \sum_{k \in M^+} x_k \Big\| + \Big\| \sum_{k \in M^-} x_k \Big\| < 2\cdot\varepsilon. \end{aligned}$$

Similarly, $\sum_{k=m}^n |\mathrm{Im}\,\langle x^*, x_k \rangle| < 2\varepsilon$. All this is clearly enough to prove $(*)$.

QED

We have now an extensive array of norm convergence conditions equivalent to unconditional summability. It is remarkable that most of these conditions are equivalent to an analogous weak convergence condition, and our next aim is to set up the transfer process by proving two subtle interchange of limits theorems, the first an old gem of Schur, its sequel the Orlicz - Pettis Theorem.

1.7 Schur's ℓ_1 Theorem: *In ℓ_1, weak convergence and norm convergence of sequences are the same.*

Proof. For the non-trivial implication there is no loss of generality in working with a weakly null sequence $(x^{(n)})$ in ℓ_1. We need to show that $\sum_k |x_k^{(n)}| \to 0$ as $n \to \infty$.

The weakly null nature of $(x^{(n)})$ ensures that for any fixed integer N, $\sum_{k=1}^N |x_k^{(n)}| \to 0$ as $n \to \infty$. The hard part is to control the tails. To achieve this, recall that B_{ℓ_∞} is weak$*$compact, and is even weak$*$metrizable, since ℓ_∞ is the dual of the separable space ℓ_1. It is perhaps worth noting that a suitable metric on B_{ℓ_∞} is given by $d(f, g) = \sum_k 2^{-k} |f_k - g_k|$. It is certainly worth recalling that a basis of weak∗neighbourhoods about some $\hat{f} \in B_{\ell_\infty}$ is given by all

$$U(\hat{f}, \delta, N) = \{f \in B_{\ell_\infty} : |f_k - \hat{f}_k| < \delta,\ 1 \le k \le N\},$$

where $\delta > 0$ and $N \in \mathbf{N}$.

The combination of the weak∗topology – which is finitary in nature – and the compact metrizability – which will enable us to invoke Baire's Category Theorem – turns out to be just what we need to gain uniform control of the tails.

Fix $\varepsilon > 0$, and for each $m \in \mathbf{N}$, set

$$B_m = \bigcap_{n \ge m} \{f \in B_{\ell_\infty} : |\langle f, x^{(n)} \rangle| \le \tfrac{\varepsilon}{3}\}.$$

Each B_m is a weak∗closed subset of B_{ℓ_∞}, since it is an intersection of such sets. The B_m's clearly grow with m, and the weak nullity of $(x^{(n)})$ gives

$$B_{\ell_\infty} = \bigcup_m B_m.$$

Baire's Category Theorem now tells us that one of the B_m's, say B_{m_0}, has non-empty interior. This ensures that we can find $\hat{f} \in B_{m_0}$, $\delta > 0$ and $N \in \mathbf{N}$ such that

$$U(\hat{f}, \delta, N) \subset B_{m_0} \subset B_m$$

for all $m \ge m_0$. By adjusting m_0 if necessary, we can also assume

$$\sum_{k=1}^{N} |x_k^{(m)}| < \frac{\varepsilon}{3}$$

for all $m \ge m_0$.

Now fix any $m \ge m_0$. Take advantage of the finitary constraints governing membership of $U(\hat{f}, \delta, N)$, and define $f \in B_{\ell_\infty}$ by

$$f_k = \hat{f}_k \text{ if } 1 \le k \le N \quad \text{and} \quad f_k = \operatorname{sign} x_k^{(m)} \text{ if } k > N.$$

Then $f \in U(\hat{f}, \delta, N) \subset B_{m_0}$, so $|\langle f, x^{(m)} \rangle| \le \varepsilon/3$. It follows that

$$\begin{aligned} \sum_k |x_k^{(m)}| &= \sum_{k \le N} |x_k^{(m)}| + \Big| \sum_k f_k x_k^{(m)} - \sum_{k \le N} f_k x_k^{(m)} \Big| \\ &\le \sum_{k \le N} (1 + |f_k|) |x_k^{(m)}| + |\langle f, x^{(m)} \rangle| \le \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

which was what we wanted. QED

With this theorem in mind we say that a Banach space has the *Schur property* if all of its weakly convergent sequences are norm convergent.

Schur's ℓ_1 Theorem is a cornerstone for our proof of the Orlicz-Pettis Theorem. We shall also make critical use of another fundamental result about weak topologies, due to S. Mazur:

- *for convex subsets, in particular subspaces, the weak and norm closures coincide.*

1.8 Orlicz-Pettis Theorem: *For sequences in a Banach space, weak subseries summability and (norm) subseries summability are the same.*

Naturally, the definition of weak subseries summability is almost a carbon copy of subseries summability: just replace the norm topology by the weak topology.

Proof. Obviously the norm property implies the weak property, but the converse requires work.

Let (x_n) be weakly subseries summable. We may as well assume that our Banach space is separable since all the action happens inside the (weakly) closed linear span of the x_n's. The proof proceeds by creating a natural operator $v : X^* \to \ell_1$ given by $v(x^*) = (\langle x^*, x_n \rangle)$. We shall use Schur's ℓ_1 Theorem to show that v is compact. From there it will only be a short step to obtain subseries summability of (x_n).

We must first justify v's existence as a bounded linear operator. Since (x_n) is weakly subseries summable, weak-$\lim_n \sum_{j=1}^n x_{k_j}$ exists for all increasing sequences (k_j) of positive integers. It follows that the *scalar* sequence $(\langle x^*, x_n \rangle)$ is unconditionally summable, so absolutely summable, for each x^* in X^*. Thus the map $v : X^* \to \ell_1$ is born. Easily, v is linear and has closed graph; hence v is a bounded linear operator.

To show that v is compact, we start with a sequence (x_m^*) in B_{X^*}. The supposed separability of X makes B_{X^*} compact and metrizable in the weak$*$topology. This allows us to extract a weak$*$convergent subsequence $(x_{m_k}^*)$ from (x_m^*) with weak$*$limit x_0^*, say. If we can show that vx_0^* is the norm limit of $(vx_{m_k}^*)$, we shall have established the compactness of v. But, since v takes values in ℓ_1, Schur's ℓ_1 Theorem is at our disposal, and so all we need to show is that vx_0^* is the weak limit of $(vx_{m_k}^*)$. For this it is enough to prove that $\langle f, vx_0^* \rangle = \lim_k \langle f, vx_{m_k}^* \rangle$ for any f in some norm dense subset of ℓ_∞. Recall that the simple functions form such a dense subset of ℓ_∞, so linearity allows us to restrict our testing set to the collection of characteristic functions $f = 1_M$ of subsets M of $\mathbf{N}$. For such an f, the weak subseries summability of (x_n) gives

$$\begin{aligned}\langle f, vx_0^* \rangle &= \sum_{n\in M} \langle x_0^*, x_n \rangle = \Big\langle x_0^*, \sum_{n\in M} x_n \Big\rangle = \lim_{k\to\infty} \Big\langle x_{m_k}^*, \sum_{n\in M} x_n \Big\rangle \\ &= \lim_{k\to\infty} \sum_{n\in M} \langle x_{m_k}^*, x_n \rangle = \lim_{k\to\infty} \langle f, vx_{m_k}^* \rangle.\end{aligned}$$

The proof could now be completed in a variety of ways. Perhaps the most elementary begins by recalling how to identify the relatively compact subsets of ℓ_1: these are the bounded subsets K with uniformly small tails, by which we mean that regardless of $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbf{N}$ such that $\sum_{n>n_\varepsilon} |a_n| \le \varepsilon$ for every $(a_n) \in K$. We shall take $v(B_{X^*})$ as our K. Then, for any finite subset M of $\mathbf{N}$ with $\min M > n_\varepsilon$,

$$\Big\| \sum_{n\in M} x_n \Big\| \le \sup_{x^*\in B_{X^*}} \sum_{n\in M} |\langle x^*, x_n \rangle| \le \varepsilon.$$

Our sequence is unordered summable. QED

Stop and think for a moment. We have actually proved more than what was stated in the Orlicz-Pettis Theorem: the compactness of the natural operator $v : X^* \to \ell_1$ is also equivalent to subseries summability of (x_n).

The Orlicz-Pettis Theorem has opened the door to the weak topology. Using the proofs of the elementary equivalences to (norm) subseries summability (1.5) as a model, it is routine to derive a similar list of weak properties equivalent to weak subseries summability. Let's draw together all that we know.

1.9 Omnibus Theorem on Unconditional Summability: *For a sequence (x_n) in a Banach space X, the following are equivalent:*

(i) *(x_n) is unconditionally summable.*
(ii) *(x_n) is unordered summable.*
(iii) *(x_n) is subseries summable.*
(iv) *(x_n) is sign summable.*
(v) *$(b_n x_n)$ is summable for every (b_n) in ℓ_∞.*
(vi) *(x_n) is weakly subseries summable.*
(vii) *(x_n) is weakly sign summable, that is $\sum_n \varepsilon_n x_n$ converges weakly in X for every choice of signs $\varepsilon_n = \pm 1$*
(viii) *$(b_n x_n)$ is weakly summable in X for every $(b_n) \in \ell_\infty$.*
(ix) *$v : X^* \to \ell_1 : x^* \mapsto (\langle x^*, x_n \rangle)_n$ is a compact operator.*
(x) *$(b_n) \mapsto \sum_n b_n x_n$ defines a compact operator $\ell_\infty \to X$.*
(xi) *$(b_n) \mapsto \sum_n b_n x_n$ defines a compact operator $c_0 \to X$.*
(xii) *$(b_n) \mapsto \sum_n b_n x_n$ defines a bounded operator $\ell_\infty \to X$.*

Our discussion has shown so far that (i) through (ix) are equivalent. Thanks to (v), the operator in (x) is just (that induced by) the adjoint of the one in (ix), and v in (ix) is the adjoint of the operator in (xi). Since (x)⇒(xi) and (x)⇒(xii)⇒(v) are trivial, the remaining equivalences are a consequence of Schauder's Theorem: a Banach space operator is compact precisely when its (bi)adjoint is.

The omission of a weak analogue of (i) from our list is inevitable. For an example, let (e_n) be the standard unit vector basis in c_0. The sequence given by $x_1 = e_1$ and $x_n = e_n - e_{n-1}$ for $n \geq 2$ shows that $\sum_n x_{\sigma(n)}$ may very well converge weakly in a Banach space for every permutation σ of N without (x_n) being unconditionally summable.

KHINCHIN'S INEQUALITY

Having seen that unconditional and absolute summability are *not* the same in general, it behoves us to seek out analytic conclusions that *can* be drawn about unconditionally summable sequences. This section serves both as an appetizer for future mathematical treats and as a provider of important ballast for times of need.

Everything hangs on our next topic, the remarkable Rademacher functions and their rôle in Khinchin's Inequality. Formally, the *Rademacher functions*

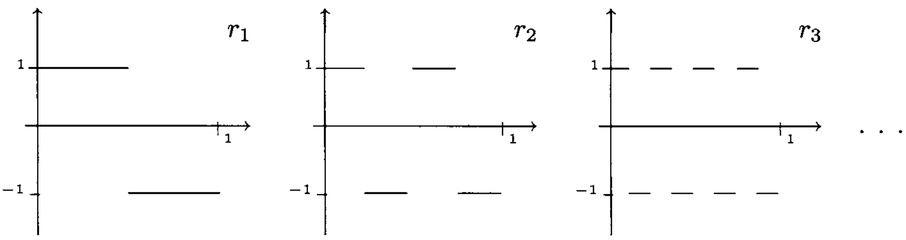
$$r_n : [0,1] \longrightarrow \mathbf{R}, \ n \in \mathbf{N},$$

are defined by setting

$$r_n(t) := \operatorname{sign}(\sin 2^n \pi t).$$

It is useful to observe that if we extend $r_1 = 1_{(0,1/2]} - 1_{[1/2,1)}$ periodically to the whole line, then $r_{n+1}(t) = r_1(2^n t)$ for all n. Occasionally, it will be convenient to refer to the constant one function as the zero'th Rademacher function r_0.

To get an understanding of the Rademacher functions, we recommend picturing their graphs rather than struggling with the formulae:



The most important feature of the Rademacher functions is that they have nice orthogonality properties. If $0 < n_1 < n_2 < \dots < n_k$ and $p_1, \dots, p_k \geq 0$ are integers, then

$$\int_0^1 r_{n_1}^{p_1}(t) \cdot \ldots \cdot r_{n_k}^{p_k}(t)\, dt = \begin{cases} 1 & \text{if each } p_j \text{ is even} \\ 0 & \text{otherwise} . \end{cases}$$

This can easily be seen from the pictures; an analytical proof would be supremely boring.

An immediate consequence is that the r_n's form an orthonormal sequence in $L_2[0,1]$, and so

$$\int_0^1 \Big| \sum a_n r_n(t) \Big|^2 dt = \sum |a_n|^2$$

for all $(a_n) \in \ell_2$. Beware! They do *not* form an orthonormal basis: $\cos(2\pi t)$ and $r_1 \cdot r_2$, for example, are orthogonal to all the r_n's.

The main result about the Rademacher functions is a powerful inequality.

1.10 Khinchin's Inequality: *For any $0 < p < \infty$, there are positive constants A_p, B_p such that regardless of the scalar sequence (a_n) in ℓ_2 we have*

$$A_p \cdot \Big(\sum_n |a_n|^2\Big)^{1/2} \leq \Big(\int_0^1 \Big|\sum_n a_n r_n(t)\Big|^p dt\Big)^{1/p} \leq B_p \cdot \Big(\sum_n |a_n|^2\Big)^{1/2}.$$