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BOUNDARY VALUE PROBLEMS FOR ELLIPTIC SYSTEMS

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Preface

This book examines the theory of boundary value problems for elliptic systems of partial differential equations, a theory which has many applications in mathematics and the physical sciences. The aim is to simplify and to algebraize the index theory by means of pseudo-differential operators and new methods in the spectral theory of matrix polynomials. This latter theory provides important tools that will enable the reader to work efficiently with the principal symbols of the elliptic and boundary operators. It also leads to important simplifications and unifications in the proofs of basic theorems such as the reformulation of the Lopatinskii condition in various equivalent forms, homotopy lifting theorems, the reduction of a system with boundary conditions to a system on the boundary, and the index formula for systems in the plane.

The book is suitable for use in graduate level courses on partial differential equations, elliptic systems, pseudo-differential operators, and matrix algebra. All the theorems are proved in detail, and the methods are well illustrated through numerous examples and exercises.

There are five parts to the book. Part I develops methods in the spectral theory of matrix polynomials which are used throughout the book; it could also be used independently as a text for a course in matrix algebra.

In Part II, there is a concise introduction to manifolds, vector bundles and differential forms. For the convenience of the reader, the development is mostly self-contained; however, it would be helpful for the reader to have had some previous acquaintance with manifold theory, and we recommend [Sp 1] for further background in the basic concepts in both the classical and modern contexts.

In Part III, pseudo-differential operators on $\mathbb{R}^n$ and on a compact manifold are studied. Chapter 7 develops the theory of pseudo-differential operators in $\mathbb{R}^n$ needed to define such operators on a manifold. Essentially, pseudo-differential operators on a manifold M are linear operators on $C^\infty(M)$ that are p.d.o.'s in local coordinates in $\mathbb{R}^n$ and satisfy a quasi-locality property. As it turns out, these operators have a main symbol (or principal symbol) defined modulo lower-order terms, and the aim of Chapter 8 is twofold: to develop the algebra of main and principal symbols, and to develop the Fredholm theory of elliptic operators in vector bundles, including the existence of a parametrix. It is beyond the scope of this book to discuss the Atiyah–Singer formula for the index of an elliptic operator on a compact manifold without boundary, but we develop, essentially, all the analytic properties required for the proof.

In the last chapter of Part III, one finds the main theorem for elliptic boundary value problems on bounded domains in $\mathbb{R}^n$: a necessary and sufficient condition for the Fredholm property to hold. An elliptic system with boundary conditions defines a Fredholm operator in appropriate Sobolev spaces if and only if the boundary operator satisfies a certain L-condition. It is assumed in the proof that the reader is familiar with the definition and basic properties of Sobolev spaces as outlined in the appendix to Chapter 7.

In Part IV, we make full use of the matrix algebra developed in Part I, and there are three important aims in this part.

First of all, a new version of the L-condition is formulated, which we refer to as the Δ-condition. This Δ-condition is expressed in terms of the boundary operator and a spectral pair for the matrix polynomial associated with the elliptic operator. In Chapter 10, we use the Δ-condition to give an elementary proof (using only matrix algebra) of the equivalence of the various formulations of the L-condition, i.e. the Lopatinskii condition, the complementing condition of Agmon, Douglis, and Nirenberg, and other conditions. Furthermore, the Δ-condition leads to an easy proof of some results of Agranovič and Dynin type, and, in addition, a homotopy lifting theorem for elliptic operators whereby a homotopy of elliptic operators is lifted to a homotopy of elliptic boundary problems. The second aim of Part IV is to show how to deform an elliptic boundary problem on Ω to a simpler form having the property that it is equivalent to an elliptic system on the double, $\tilde{\Omega}$, of Ω. The double is a compact manifold, without boundary, to which the Atiyah–Singer theory can be applied. Finally, in Part IV there is a broad discussion of the transmission property for pseudo-differential operators, and another proof of the main theorem for elliptic boundary problems. This second proof uses the Calderón operator to construct a parametrix, i.e. an operator which inverts the boundary value problem, modulo an integral operator with C^∞ kernel. We follow [Hö 3] in the construction of a parametrix by a method inspired by the classical integral representation of solutions of a boundary value problem for the Laplace operator in terms of single- and double-layer potentials (see the introduction to Chapter 14).

Part V is devoted to elliptic boundary problems on bounded domains in the plane. The aim of this part is to prove the index formula, i.e. that the topological index is equal to the analytical index. We study in further detail the L-condition for differential operators in the plane, then define the topological index of an elliptic boundary problem. The proof of the index formula relies on the homotopy lifting theorem mentioned above, in order to reduce an elliptic boundary value problem to a type of Riemann–Hilbert problem for which case there is a well-known formula for the index. With the exception of §16.5, we consider only differential operators in Part V, and the proofs here do not require the use of pseudo-differential operators. In the last chapter, a homotopy classification of 2×2 systems in the plane is accomplished and several examples are studied in detail.

Index of Notation

Transpose of a matrix, $[a_{ij}]^T = [a_{ji}]$
Transpose of a linear map, ${}^t f$
Inner product of column vectors in $\mathbb{R}^n$, $(a, b)_{\mathbb{R}^n} = b^T a = \sum a_i b_i$
Hermitian adjoint of a complex matrix, $[a_{ij}]^h = [\bar{a}_{ji}]$
Inner product of column vectors in $\mathbb{C}^n$, $(a, b)_{\mathbb{C}^n} = b^h a = \sum a_i \bar{b}_i$
Kernel of a linear map, ker A
Image of a linear map, im A

Miscellaneous Notes

In Part II, finite dimensional vector spaces are denoted by bold letters **E**, **F**, etc., and the space of linear maps **E** → **F** is denoted by $L(\mathbf{E}, \mathbf{F})$. Elsewhere in the book, finite dimensional vector spaces are denoted by German letters $\mathfrak{M}$, $\mathfrak{N}$, etc., and the space of linear maps $\mathfrak{M} \to \mathfrak{N}$ is denoted by $\mathscr{L}(\mathfrak{M}, \mathfrak{N})$.

Throughout the book, ℓ denotes the degree of the matrix polynomial $L(\lambda) = \sum_{j=0}^{l} A_j \lambda^j$. The letter ℓ is also used occasionally to label the Sobolev spaces, for instance, as in Theorem 9.32.