Part I

A Spectral Theory of Matrix Polynomials

A matrix polynomial is a polynomial function of a complex variable of the form $L(\lambda) = \sum_{j=0}^{l} A_j \lambda^j$ where the coefficients, A_j, are $p \times p$ matrices of complex numbers. Basic matrix theory (including the spectral theory, Jordan form, etc.) is a theory of matrix polynomials $I\lambda - A$ of first degree. The purpose of Part I is to develop a spectral theory for matrix polynomials of arbitrary degree.

We always assume that $L(\lambda)$ is a *regular* matrix polynomial, i.e. $\det L(\lambda) \not\equiv 0$. The spectrum of $L(\lambda)$ is the set of all complex numbers λ such that $L(\lambda)$ is not invertible,

$$\mathrm{sp}(L) = \{\lambda; \det L(\lambda) = 0\}.$$

For a matrix polynomial of degree one, $I\lambda - A$, we write $\mathrm{sp}(A)$ rather than $\mathrm{sp}(I\lambda - A)$ in order to be consistent with the usual definition of the spectrum of the matrix A, that is,

$$\mathrm{sp}(A) = \{\lambda; \det(I\lambda - A) = 0\}.$$

Chapters 1 and 2 develop the spectral theory for the general case of any regular matrix polynomial $L(\lambda)$. Chapter 3 treats the *monic* case, where the leading coefficient, A_l, is the identity matrix. Chapter 4 is needed for technical reasons in Chapter 15.

The spectral triples introduced in Chapter 2 lead to a method for understanding the Lopatinskii condition for elliptic boundary problems, and for reformulating it in various ways as needed (see §10.1). The application of this spectral theory to partial differential equations is new, so we have made the presentation of Part I essentially self-contained, depending only on a knowledge of basic matrix theory and complex analysis. There is, of course, a close relationship between the spectral theory of matrix polynomials, $L(\lambda)$, and the solution space of the equation

$$L(d/dt)u = 0.$$

Due to our interest in differential equations, we do not hesitate to take advantage of this relationship in order to motivate or simplify proofs whenever possible.

1
Matrix polynomials

Let α denote the degree of det $L(\lambda)$. In this chapter we show how the spectral data for the matrix polynomial

$$L(\lambda) = \sum_{j=0}^{l} A_j \lambda^j$$

can be organized in a pair of matrices (X, J), where J is an $\alpha \times \alpha$ Jordan matrix and the columns of the $p \times \alpha$ matrix X are made up of various Jordan chains for each eigenvalue of $L(\lambda)$. There may of course be more than one Jordan chain corresponding to the same eigenvalue.

The idea is as follows. Let $\lambda_0 \in \mathrm{sp}(L)$ be an eigenvalue, and let μ denote the multiplicity of λ_0 as a root of det $L(\lambda) = 0$. Suppose we have found a set of m Jordan chains (see §1.2 for definitions)

$$x_0^{(i)}, \ldots, x_{\mu_i - 1}^{(i)} \qquad i = 1, \ldots, m$$

for $L(\lambda)$ corresponding to the eigenvalue λ_0, such that the eigenvectors $x_0^{(1)}, \ldots, x_0^{(m)}$ (the first vector in each Jordan chain) are linearly independent and such that $\Sigma \mu_i = \mu$. Then introduce a $p \times \mu$ matrix X_0 and a $\mu \times \mu$ block diagonal Jordan matrix J_0 as follows:

$$X_0 = [x_0^{(1)} \cdots x_{\mu_1 - 1}^{(1)} | \cdots | x_0^{(m)} \cdots x_{\mu_m - 1}^{(m)}], \qquad J_0 = \mathrm{diag}(J_0^{(i)})_{i=1}^{m}.$$

The columns of X_0 are the vectors in the given Jordan chains, and the $J_0^{(i)}$ are Jordan blocks of size $\mu_i \times \mu_i$ with eigenvalue λ_0. It can be shown that the pair (X_0, J_0) satisfies the properties

$$\text{(a)} \ \sum_{j=0}^{l} A_j X_0 J_0^j = 0, \qquad \text{(b)} \ \ker \begin{pmatrix} X_0 \\ X_0 J_0 \\ \vdots \\ X_0 J_0^{l-1} \end{pmatrix} = 0,$$

and we call (X_0, J_0) a *Jordan pair* of $L(\lambda)$ corresponding to the eigenvalue λ_0. Now, if we have a set of Jordan pairs $(X_1, J_1), \ldots, (X_s, J_s)$ corresponding to the distinct eigenvalues $\lambda_1, \ldots, \lambda_s$ of $L(\lambda)$. we let

$$X = [X_1 \cdots X_s], \qquad J = \mathrm{diag}(J_k)_{k=1}^{s},$$

and it can be shown that the pair (X, J) also has the properties (a), (b). We call (X, J) a *finite Jordan pair* for $L(\lambda)$. Note that X is a $p \times \alpha$ matrix and J is $\alpha \times \alpha$, where α is the degree of det $L(\lambda)$.

By virtue of the properties (a), (b) we shall see that *every* solution of the equation $L(d/dt)u = 0$ can be written in the form

$$u(t) = Xe^{tJ}c$$

for a unique $c \in \mathbb{C}^\alpha$.

It turns out that the columns of the matrix X form a *canonical* set of Jordan chains for $L(\lambda)$ in the sense of Definitions 1.24, 1.27. The main result of this chapter is the proof that a canonical set of Jordan chains does exist.

For technical reasons, however, the order of development of the theory in Chapter 1 is somewhat different from what was indicated above. In the first section, §1.1, we show by row and column operations that any matrix polynomial $L(\lambda)$ can be transformed to a diagonal form $D(\lambda)$. In §1.2 it is proved that the length of a Jordan chain is preserved in the transformation from $L(\lambda)$ to $D(\lambda)$, which provides valuable information since the structure of the Jordan chains for a diagonal matrix polynomial is easy to establish. In §1.3 we define the notion of a *partial spectral pair* (X, T) – an admissible pair satisfying properties like (a), (b) but with T not necessarily a Jordan matrix and with no condition on the spectrum of T – and then prove some preliminary results which follow formally from this definition. In §1.4 the existence of a canonical set of Jordan chains is proved.

1.1 Smith canonical form

We begin by showing that any matrix polynomial $L(\lambda)$ can be reduced to upper triangular form (zeros below the diagonal) by means of elementary row operations. The types of operations permitted are the following:

(a) the interchange of two rows: $R_i \leftrightarrow R_j$,
(b) the addition to some row of another row multiplied by any polynomial in λ: $p(\lambda)R_j + R_i$,
(c) the multiplication of a row by a nonzero constant: cR_i.

A matrix polynomial obtained from the identity matrix I by means of one such row operation is referred to as an *elementary* matrix polynomial. Note that the determinant of such a matrix polynomial is a nonzero constant.

Two matrix polynomials, $L(\lambda)$ and $M(\lambda)$, are said to be *row equivalent* if there exists a finite sequence of row operations which transform $L(\lambda)$ to $M(\lambda)$ (or vice versa), i.e.

$$M(\lambda) = E(\lambda)L(\lambda),$$

where $E(\lambda)$ is a product of elementary matrix polynomials (and hence det $E(\lambda)$ is a nonzero constant). Note that row equivalence is an equivalence relation between matrix polynomials.

Proposition 1.1 *Every $p \times p$ matrix polynomial $L(\lambda)$ is row equivalent to a matrix polynomial in upper triangular form.*

Proof We may assume that the first column of $L(\lambda)$ is not identically 0. Let $a(\lambda)$ be the monic polynomial of least degree among all first-column entries of all matrix polynomials row equivalent to $L(\lambda)$, and let $M(\lambda) = [m_{ij}(\lambda)]$ be a matrix polynomial row equivalent to $L(\lambda)$ with $m_{11}(\lambda) = a(\lambda)$. The claim is that each entry in the first column of $M(\lambda)$ is necessarily divisible by $a(\lambda)$. Indeed, consider any entry $b(\lambda)$ in the first column of $M(\lambda)$, say in the $(i, 1)$ position. By the Euclidean algorithm for scalar polynomials $b(\lambda) = q(\lambda)a(\lambda) + r(\lambda)$, where $r(\lambda) \equiv 0$ or $\deg r(\lambda) < \deg a(\lambda)$. Now, the row operation $-q(\lambda)R_1 + R_i$ leads to a matrix polynomial row equivalent to $L(\lambda)$ having $r(\lambda)$ in the $(i, 1)$ position; therefore, by the way $a(\lambda)$ was chosen, it must be true that $r(\lambda) \equiv 0$. Performing the row operations $-b(\lambda)/a(\lambda)R_1 + R_i$, we obtain that $L(\lambda)$ is row equivalent to a matrix polynomial of the form

$$\left(\begin{array}{c|c} a(\lambda) & * \\ \hline 0 & L_1(\lambda) \end{array}\right),$$

where $L_1(\lambda)$ is a $(p-1) \times (p-1)$ matrix polynomial. Repeated application of this method leads to a proof of the proposition by induction on the dimension p of the matrix polynomial.

Corollary *Let $L(\lambda)$ be a $p \times p$ matrix polynomial with* $\det L(\lambda) \not\equiv 0$ *and let α denote the degree of* $\det L(\lambda)$. *Then the dimension of the solution space of the homogeneous differential equation*

$$L\left(\frac{d}{dt}\right)u(t) = 0 \tag{1}$$

is equal to α. Here $u(t) = [u_1(t) \cdots u_p(t)]^T$ is a $\mathbb{C}^p$-valued function.

Proof Let $M(\lambda) = [m_{ij}(\lambda)]$ be an upper triangular matrix polynomial obtained from $L(\lambda)$ by elementary row operations. The new system of equations

$$M\left(\frac{d}{dt}\right)u(t) = 0 \tag{2}$$

is equivalent to the original, since each such operation is reversible. Let l_i be the degree of $m_{ii}(\lambda)$ $(i = 1, \ldots, p)$. The pth equation is a scalar equation $m_{pp}(d/dt)u_p(t) = 0$ with l_p-dimensional solution space. Then by induction on the number p of equations, it is easily seen that the solution space of (2) (and hence (1)) has dimension $\sum_{i=1}^{p} l_i = \deg \det L(\lambda)$.

We will also need the fact that any matrix polynomial can be transformed by row and column operations to *diagonal* form. The column operations that are permitted are of the same three types as stated earlier for row operations, and there is a corresponding definition of column equivalence.

Also, we shall say that $L(\lambda)$ and $M(\lambda)$ are *equivalent under row and column operations* if there is a finite sequence of row or column operations that transforms $M(\lambda)$ to $L(\lambda)$, i.e.

$$M(\lambda) = E(\lambda)L(\lambda)F(\lambda),$$

where $E(\lambda)$ and $F(\lambda)$ are products of elementary matrix polynomials.

Proposition 1.2 *Let $L(\lambda)$ be a $p \times p$ matrix polynomial. Then $L(\lambda)$ is equivalent under row and column operations to a matrix polynomial of the form*

$$\tilde{L}(\lambda) = \left(\begin{array}{c|c} a(\lambda) & 0 \cdots 0 \\ \hline 0 & L_1(\lambda) \end{array}\right), \tag{3}$$

where $L_1(\lambda)$ is a $(p-1) \times (p-1)$ matrix polynomial and each entry of $L_1(\lambda)$ is divisible by $a(\lambda)$.

Proof We may assume that $L(\lambda) \not\equiv 0$. Let $a(\lambda)$ be the monic polynomial of least degree among all entries of all matrix polynomials row equivalent to $L(\lambda)$, and let $M(\lambda) = [m_{ij}(\lambda)]$ be a matrix polynomial row equivalent to $L(\lambda)$ with $m_{11}(\lambda) = a(\lambda)$. As in the proof of Proposition 1.1, each entry in the first column of $M(\lambda)$ is necessarily divisible by $a(\lambda)$. Since we are now permitting column operations, the same is true of each entry in the first row of $M(\lambda)$. By row and column operations, we then obtain that $L(\lambda)$ is equivalent to a matrix polynomial of the form (3). The claim is that each entry of $L_1(\lambda)$ must be divisible by $a(\lambda)$. Indeed, consider any entry $b(\lambda)$ of $L_1(\lambda)$, say in the (i, j) position of $\tilde{L}(\lambda)$ $(i > 1,\ j > 1)$. Then $b(\lambda) = q(\lambda)a(\lambda) + r(\lambda)$, where $r(\lambda) \equiv 0$ or $\deg r(\lambda) < \deg a(\lambda)$. By performing the row operation $R_i + R_1$ and the column operation $-q(\lambda)C_1 + C_j$ on $\tilde{L}(\lambda)$, we obtain a matrix polynomial that has $r(\lambda)$ in the $(1, j)$ position. Since this matrix polynomial is equivalent to $L(\lambda)$, then, by the way $a(\lambda)$ was chosen, we must have $r(\lambda) \equiv 0$. Hence $b(\lambda)$ is divisible by $a(\lambda)$. This completes the proof of the proposition.

Corollary *Repeated application of Proposition* 1.2 *leads to the following result: Every matrix polynomial $L(\lambda)$ is equivalent to a diagonal matrix polynomial*

$$D(\lambda) = \begin{pmatrix} d_1(\lambda) & & & & & \\ & \ddots & & & & \\ & & d_s(\lambda) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix},$$

with monic scalar polynomials $d_i(\lambda)$ such that $d_i(\lambda)$ is divisible by $d_{i-1}(\lambda)$.

The scalar polynomials $d_i(\lambda)$ in the preceding corollary are uniquely determined by $L(\lambda)$ (see Theorem 1.4); $D(\lambda)$ is called the *Smith canonical form* of $L(\lambda)$. Note that if $\det L(\lambda) \not\equiv 0$ then $D(\lambda)$ has no zero entries on the diagonal, that is, $s = p$.

Let $L(\lambda) = [a_{ij}(\lambda)]$ be a $p \times p$ matrix polynomial, and choose k rows $1 \leqslant i_1 < \cdots < i_k \leqslant p$ and k columns $1 \leqslant j_1 < \cdots < j_k \leqslant p$ from $L(\lambda)$. Then the determinant

$$\det[a_{i_m j_n}(\lambda)]_{m,n=1}^{k}$$

is called a $k \times k$ subdeterminant, or minor, of $L(\lambda)$. We let $\phi_k(\lambda)$ denote the monic greatest common divisor (gcd) of all nonzero $k \times k$ subdeterminants of $L(\lambda)$ and let $\phi_k(\lambda) \equiv 0$ if all the subdeterminants of order k are zero. Also let $\phi_0(\lambda) = 1$.

Lemma 1.3 *The polynomials $\phi_k(\lambda)$ are invariant under row and column operations. That is, if $L(\lambda)$ and $M(\lambda)$ are equivalent matrix polynomials, then $\phi_k^M(\lambda) = \phi_k^L(\lambda)$, where $\phi_k^M(\lambda)$, $\phi_k^L(\lambda)$ denote the monic gcd of $k \times k$ subdeterminants of $M(\lambda)$, $L(\lambda)$, respectively.*

Proof First we prove the lemma under the assumption that $M(\lambda)$ is obtained from $L(\lambda)$ by a single row operation. The rows of $M(\lambda)$ are linear combinations with coefficients in $\mathbb{C}[\lambda]$ of the rows of $L(\lambda)$. Due to the multilinearity (i.e. linearity in each row) of the determinant function, it follows that each $k \times k$ subdeterminant of $M(\lambda)$ is a linear combination of the $k \times k$ subdeterminants of $L(\lambda)$. Consequently, $\phi_k^M(\lambda)$ is divisible by $\phi_k^L(\lambda)$. But if we consider the inverse row operation transforming $L(\lambda)$ to $M(\lambda)$, it follows that $\phi_k^L(\lambda)$ is divisible by $\phi_k^M(\lambda)$, and since both are monic, that $\phi_k^M(\lambda) = \phi_k^L(\lambda)$. The same proof holds if $M(\lambda)$ is obtained from $L(\lambda)$ by a single column operation, and then it clearly holds for any number of row and column operations.

Theorem 1.4 *Every matrix polynomial is equivalent under row and column operations to a unique diagonal matrix polynomial*

$$D(\lambda) = \begin{pmatrix} d_1(\lambda) & & & & & \\ & \ddots & & & & \\ & & d_s(\lambda) & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, \tag{4}$$

with monic scalar polynomials $d_i(\lambda)$ such that $d_i(\lambda)$ is divisible by $d_{i-1}(\lambda)$. The diagonal entries $d_i(\lambda)$ are given by the following formulas:

$$d_i(\lambda) = \frac{\phi_i(\lambda)}{\phi_{i-1}(\lambda)} \qquad i = 1, \ldots, s \tag{5}$$

Proof The existence of the diagonal form has been proved in Proposition 1.2. Suppose now that $L(\lambda)$ is equivalent to a diagonal matrix polynomial (4) where the $d_i(\lambda)$ are monic scalar polynomials such that $d_i(\lambda)$ is divisible by $d_{i-1}(\lambda)$. By Lemma 1.3, it follows that

$$\begin{aligned} \phi_k(\lambda) &= \text{gcd of } k \times k \text{ subdeterminants of } D(\lambda) \\ &= d_1(\lambda) \cdots d_k(\lambda), \end{aligned}$$

and the formulas (5) follow immediately, proving uniqueness of the Smith canonical form.

1.2 Eigenvectors and Jordan chains

A non-zero column vector $x_0 \in \mathbb{C}^p$ such that $L(\lambda_0)x_0 = 0$ is said to be an *eigenvector* of the matrix polynomial $L(\lambda)$ corresponding to the eigenvalue λ_0. It follows that $\det L(\lambda_0) = 0$. The *spectrum* of $L(\lambda)$, denoted sp(L), is the set of $\lambda \in \mathbb{C}$ such that $\det L(\lambda) = 0$. From now on, it is always assumed that $\det L(\lambda) \not\equiv 0$, so that sp($L$) consists of a finite number of eigenvalues.

We seek solutions of (1) in the form of vector-valued functions

$$u(t) = \left[\frac{t^{k-1}}{(k-1)!}x_0 + \frac{t^{k-2}}{(k-2)!}x_1 + \cdots + x_{k-1}\right]e^{\lambda_0 t} \tag{6}$$

where $x_j \in \mathbb{C}^p$ and $x_0 \neq 0$. It is not hard to show that x_0 must be an eigenvector of $L(\lambda)$ corresponding to λ_0. In fact we have the following proposition.

Proposition 1.5 *The vector-valued function $u(t)$ given by* (6) *is a solution of* (1) *if and only if*

$$\sum_{j=0}^{i} \frac{1}{j!} L^{(j)}(\lambda_0)x_{i-j} = 0 \qquad i = 0, \ldots, k-1 \tag{7}$$

Proof The Taylor series for $L(\lambda)$ about λ_0 is

$$L(\lambda) = L(\lambda_0) + L'(\lambda_0)\cdot(\lambda - \lambda_0) + \cdots + \frac{1}{l!}L^{(l)}(\lambda_0)\cdot(\lambda-\lambda_0)^l$$

Then, replacing λ by d/dt, we obtain

$$L\left(\frac{d}{dt}\right)u(t) = L(\lambda_0)u(t) + L'(\lambda_0)\cdot\left(\frac{d}{dt} - \lambda_0\right)u(t) + \cdots + \frac{1}{l!}L^{(l)}(\lambda_0) \times \left(\frac{d}{dt} - \lambda_0\right)^l u(t) \tag{8}$$

Computation shows that

$$\left(\frac{d}{dt} - \lambda_0\right)^j u(t) = \left[\frac{t^{k-j-1}}{(k-j-1)!}x_0 + \frac{t^{k-j-2}}{(k-j-2)!}x_1 + \cdots + x_{k-j-1}\right]e^{\lambda_0 t}$$

for $j = 0, \ldots, k-1$ and $(d/dt - \lambda_0)^j u(t) = 0$ for $j = k, k+1, \ldots,$ and by substitution of these results in the formula (8), the equalities (7) are obtained.

A sequence of p-dimensional column vectors $x_0, x_1, \ldots, x_{k-1}$, where $x_0 \neq 0$, for which equalities (7) hold is called a *Jordan chain of length k for* $L(\lambda)$ corresponding to the eigenvalue λ_0. The vectors $x_1, \ldots, x_{k-1}$ are sometimes known as generalized eigenvectors.

We shall say that an eigenvector x_0 of $L(\lambda)$ corresponding to λ_0 is of *rank*

k if the maximal length of a Jordan chain of $L(\lambda)$ corresponding to λ_0 with x_0 as eigenvector is k. Note that if $x_0, x_1, \ldots, x_{k-1}$ is a Jordan chain for $L(\lambda)$, then so is $x_0, x_1, \ldots, x_{s-1}$ for any $s \leqslant k$.

The definition of a Jordan chain of a matrix polynomial is a generalization of the usual idea of Jordan chains for a matrix T. Indeed, let $v_0, v_1, \ldots, v_{k-1}$ be a Jordan chain for T, that is, $Tv_0 = \lambda_0 v_0$, $Tv_1 = \lambda_0 v_1 + v_0, \ldots, Tv_{k-1} = \lambda_0 v_{k-1} + v_{k-2}$. Then these equalities mean exactly that $v_0, v_1, \ldots, v_{k-1}$ is a Jordan chain for the matrix polynomial $I\lambda - T$.

For a matrix polynomial of degree one, $I\lambda - T$, the vectors in a Jordan chain are linearly independent. This is not true for a matrix polynomial of degree greater than one. The following examples show that even the zero vector is admissible as a generalized eigenvector.

Example 1.6

$$L(\lambda) = \begin{pmatrix} (\lambda-2)^2 & -\lambda+2 \\ 0 & (\lambda-2)^2 \end{pmatrix}$$

Since $\det L(\lambda) = (\lambda - 2)^4$, there is one eigenvalue, namely, $\lambda_0 = 2$. Every non-zero vector in $\mathbb{C}^2$ is an eigenvector of $L(\lambda)$ corresponding to λ_0. It is easy to show that all Jordan chains of $L(\lambda)$ can be described as follows:

(1) Jordan chains of length 1 are $x_0 = \begin{pmatrix} a \\ b \end{pmatrix}$, where $a, b \in \mathbb{C}$ are not both zero.

(2) Jordan chains of length 2 are $x_0 = \begin{pmatrix} a \\ b \end{pmatrix}$, x_1, where $a \neq 0$ and $x_1 \in \mathbb{C}^2$ is arbitrary.

(3) Jordan chains of length 3 are $x_0 = \begin{pmatrix} a \\ 0 \end{pmatrix}$, $x_1 = \begin{pmatrix} b \\ a \end{pmatrix}$, x_2, where $a \neq 0$ and $b \in \mathbb{C}$, $x_2 \in \mathbb{C}^2$ are arbitrary.

There are no Jordan chains of length $\geqslant 4$.

Example 1.7 Let $L(\lambda)$ be a scalar polynomial ($p = 1$). Let $L(\lambda_0) = 0$, and let k be the multiplicity of the root λ_0. Then any sequence of complex numbers $x_0, x_1, \ldots, x_{k-1}$, where $x_0 \neq 0$, is a Jordan chain of maximal length of $L(\lambda)$ corresponding to λ_0. There are no Jordan chains of length $> k$ since $L^{(j)}(\lambda_0) = 0$ for $j = 0, \ldots, k-1$ but $L^{(k)}(\lambda_0) \neq 0$.

Example 1.8 Let $L(\lambda) = \operatorname{diag}(d_i(\lambda))_{i=1}^{p}$ be a diagonal matrix polynomial, where the $d_i(\lambda)$ are scalar polynomials. Let $\det L(\lambda_0) = 0$ and let κ_i be the multiplicity of λ_0 as a root of $d_i(\lambda) = 0$. Also, denote by e_i the ith coordinate vector in $\mathbb{C}^p$. Then it is easily verified that the following are Jordan chains of maximal length for $L(\lambda)$ for any $i = 1, \ldots, p$ such that $\kappa_i \neq 0$:

$$c_0 e_i, \ldots, c_{\kappa_i - 1} e_i,$$

where $c_j \in \mathbb{C}$ and $c_0 \neq 0$. Hence the eigenvector e_i has rank κ_i. More generally, if $x_0 = [x_0^{(i)}]_{i=1}^p$ is any eigenvector of $L(\lambda)$ corresponding to λ_0, then the rank of x_0 is equal to $\min\{\kappa_i; x_0^{(i)} \neq 0\}$.

The next goal will be to characterize the lengths of Jordan chains of maximal length (Proposition 1.10), and then to construct a canonical set of Jordan chains (§1.4).

Lemma 1.9 *Let $M(\lambda)$ be a $p \times p$ matrix polynomial and let $E(\lambda)$ and $F(\lambda)$ be $p \times p$ matrix polynomials with constant nonzero determinants. Then $x_0, \ldots, x_{k-1}$ is a Jordan chain of the matrix polynomial $L(\lambda) := E(\lambda)M(\lambda)F(\lambda)$ corresponding to some $\lambda_0 \in \mathrm{sp}(L)$ if and only if the vectors*

$$y_j = \sum_{i=0}^{j} \frac{1}{i!} F^{(i)}(\lambda_0) x_{j-i} \qquad j = 0, \ldots, k-1$$

form a Jordan chain of $M(\lambda)$ corresponding to λ_0.

Proof Let $x_0, \ldots, x_{k-1}$ *be a Jordan chain of* $L(\lambda)$ corresponding to λ_0 and define $u(t)$ as in (6). Then, as in Proposition 1.5, we have $L(d/dt)u(t) = 0$ so that

$$E\left(\frac{d}{dt}\right)M\left(\frac{d}{dt}\right)F\left(\frac{d}{dt}\right)u(t) = 0 \tag{9}$$

Since $E^{-1}(\lambda)$ is also a matrix polynomial, we can operate with $E^{-1}(d/dt)$ on both sides of (9) to obtain

$$M\left(\frac{d}{dt}\right)v(t) = 0, \tag{10}$$

where $v(t) = F(d/dt)u(t)$. Then, writing

$$F(\lambda) = \sum_{j=0}^{\deg F} \frac{1}{j!} F^{(j)}(\lambda_0)(\lambda - \lambda_0)^j,$$

it follows that

$$v(t) = \left[\frac{t^{k-1}}{(k-1)!} y_0 + \frac{t^{k-2}}{(k-2)!} y_1 + \cdots + y_{k-1}\right] e^{\lambda_0 t} \tag{11}$$

where

$$y_i = \sum_{j=0}^{i} \frac{1}{j!} F^{(j)}(\lambda_0) x_{i-j} \qquad i = 0, \ldots, k-1. \tag{12}$$

Due to (10), (11) and Proposition 1.5, we see that $y_0, \ldots, y_{k-1}$ is a Jordan chain of $M(\lambda)$ corresponding to λ_0. Note that $y_0 \neq 0$ since $\det F(\lambda_0) \neq 0$. The converse is obviously true since $M(\lambda) = E^{-1}(\lambda)L(\lambda)F^{-1}(\lambda)$, where $E^{-1}(\lambda)$ and $F^{-1}(\lambda)$ are matrix polynomials with constant non-zero determinants.

Let $D(\lambda) = \mathrm{diag}(d_i(\lambda))_{i=1}^p$ denote the Smith canonical form of $L(\lambda)$. If $\det L(\lambda_0) = 0$, let κ_i denote the multiplicity of λ_0 as a zero of $d_i(\lambda)$. Then $\kappa_1 \leqslant \cdots \leqslant \kappa_p$ are known as the *partial multiplicities* of $L(\lambda)$ at λ_0. Note that

$$\sum_{i=1}^{p} \kappa_i = \alpha_0,$$

the multiplicity of λ_0 as a zero of det $L(\lambda)$.

Proposition 1.10 *The lengths of Jordan chains of $L(\lambda)$ of maximal length are exactly the non-zero partial multiplicities of $L(\lambda)$ corresponding to λ_0.*

Proof By Theorem 1.4, there exist matrix polynomials $E(\lambda)$ and $F(\lambda)$ with constant non-zero determinant such that $L(\lambda) = E(\lambda)D(\lambda)F(\lambda)$, where $D(\lambda)$ is the Smith canonical form of $L(\lambda)$. Then the equations (12) of Lemma 1.9 define a one-to-one correspondence between Jordan chains of $L(\lambda)$ and Jordan chains of $D(\lambda)$ corresponding to λ_0 – the inverse to (12) is found by using the coefficients of the matrix polynomial $F^{-1}(\lambda)$ – that preserves the length of Jordan chains. Since $D(\lambda)$ is a diagonal matrix, then the lengths of Jordan chains of $D(\lambda)$ – and hence $L(\lambda)$ – of maximal length are simply the non-zero partial multiplicities of $L(\lambda)$ corresponding to λ_0 (see Example 1.8).

Remark It follows from Proposition 1.10 that the rank of any eigenvector corresponding to λ_0 does not exceed the multiplicity of λ_0 as a zero of det $L(\lambda)$.

The next proposition introduces a convenient way to summarize the information given in a Jordan chain.

Proposition 1.11 *The vectors $x_0, x_1, \ldots, x_{k-1}$ form a Jordan chain corresponding to λ_0 for the $p \times p$ matrix polynomial $L(\lambda) = \sum_{j=0}^{l} A_j\lambda^j$ if and only if $x_0 \neq 0$ and*

$$A_0X_0 + A_1X_0J_0 + \cdots + A_lX_0J_0^l = 0 \tag{13}$$

where $X_0 = [x_0 \cdots x_{k-1}]$ is a $p \times k$ matrix, and

$$J_0 = \begin{pmatrix} \lambda_0 & 1 & & & \\ & \lambda_0 & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & \lambda_0 \end{pmatrix} \tag{14}$$

is the $k \times k$ Jordan block with eigenvalue λ_0.

Proof Since

$$e^{tJ_0} = \begin{pmatrix} 1 & t & \cdots & \dfrac{t^{k-1}}{(k-1)!} \\ & 1 & \cdot & \\ & & \cdot & \cdot \\ & & & t \\ & & & 1 \end{pmatrix} e^{t\lambda_0} \tag{15}$$