1

The binomial coefficients

The idea of choosing a number of items is going to occur many times in any course on combinatorics and so an appropriate first chapter is on selections and on the number of ways those selections can be made. In this chapter there are no theorems, simply a range of miscellaneous ideas presented as examples. Although the ideas are elementary and although many of them will be familiar, the chances are that most readers will find some unfamiliar aspect amongst them.

Given n objects let $\binom{n}{k}$ denote the number of different selections of k objects from the n: the order in which the k objects are chosen is irrelevant.

Example *There are 16 chapters in this book: how many different selections of two chapters are there? In general, in how many ways can two be chosen from n?*

Solution

$$\binom{16}{2} = \frac{16 \times 15}{2} = 120.$$

There are several ways of seeing this. For example you could list all pairs of chapters, with 16 choices for the first of the pair and 15 remaining choices for the other member of the pair:

1, 2 1, 3 1, 4 ⋮ 1, 16 2, 1 2, 3 2, 4 ⋮ 16, 14 16, 15	There are clearly 16 × 15 pairs in this list but note that any two chapters appear in the list as two different pairs. For example 7 and 11 will appear once as '7, 11' and then again as '11, 7'. Since the order is irrelevant to us we only want to count half of this list, making $\binom{16}{2}$ equal to 120 as above.

Another way of seeing that $\binom{16}{2} = 120$ is by again listing the pairs but avoiding a repeated pair by only writing down those (like '7, 11') where the first member of the pair is the lower:

$$\left.\begin{matrix}1,2\\1,3\\\vdots\\1,16\end{matrix}\right\}15 \quad \left.\begin{matrix}2,3\\\vdots\\2,16\end{matrix}\right\}14 \quad \left.\begin{matrix}3,4\\\vdots\\3,16\end{matrix}\right\}13 \quad \begin{matrix}\vdots & \vdots\end{matrix} \quad \left.\begin{matrix}14,15\\14,16\end{matrix}\right\}2 \quad 15,16\}\ 1$$

In this list each possible pair occurs exactly once and so we must simply count the number of pairs in the list to give

$$\binom{16}{2} = 15 + 14 + 13 + \cdots + 2 + 1 = \tfrac{1}{2} \times 15 \times 16$$

as before.

Clearly both these arguments extend to choosing two from n to give

$$\binom{n}{2} = \tfrac{1}{2}n(n-1). \qquad \square$$

Example *In how many ways can three of the 16 chapters be chosen? Show that in general*

$$\binom{n}{3} = \binom{n-1}{2} + \binom{n-2}{2} + \binom{n-3}{2} + \cdots + \binom{3}{2} + \binom{2}{2}.$$

Solution Following the first of the methods in the previous solution gives a total list of $16 \times 15 \times 14$ triples of chapters; 16 choices for the first of the triple, 15 remaining choices for the second and 14 remaining choices for the third. But any particular set of three chapters will be in that long list six times (for example the set of three chapters 7, 11 and 15 will occur in the list as '7, 11, 15', '7, 15, 11', '11, 7, 15', '11, 15, 7', '15, 7, 11' and '15, 11, 7'). So the number of different selections of three chapters from the 16 is

$$\binom{16}{3} = \frac{16 \times 15 \times 14}{6} = 560.$$

The second method in the previous solution was to list the required selections with the lowest first to avoid repeated pairs. In a similar way we could now list the triples with each triple in increasing order: to count the number of triples in this list we can use the result of the previous example:

$$\left.\begin{array}{c}1,2,3\\1,2,4\\ \vdots\\1,15,16\end{array}\right\}\quad \binom{15}{2} \text{ of the triples begin with a 1,}$$

$$\left.\begin{array}{c}2,3,4\\ \vdots\\2,15,16\end{array}\right\}\quad \binom{14}{2} \text{ of the triples begin with a 2,}$$

$$\left.\begin{array}{c}3,4,5\\ \vdots\\3,15,16\end{array}\right\}\quad \binom{13}{2} \text{ of the triples begin with a 3,}$$

$$\vdots \qquad\qquad \vdots$$

$$\left.\begin{array}{c}13,14,15\\13,14,15\\13,15,16\end{array}\right\}\quad \binom{3}{2} \text{ of the triples begin with a 13,}$$

$$14,15,16\} \quad \binom{2}{2} \text{ of the triples begin with a 14.}$$

Hence

$$\binom{16}{3} = \binom{15}{2} + \binom{14}{2} + \binom{13}{2} + \cdots + \binom{3}{2} + \binom{2}{2} = 560.$$

It is left to the reader to show that a similar argument choosing three items from n would yield the relationship

$$\binom{n}{3} = \binom{n-1}{2} + \binom{n-2}{2} + \binom{n-3}{2} + \cdots + \binom{3}{2} + \binom{2}{2}. \qquad \square$$

That gentle introduction leads to the idea that if k items are chosen from n, where the order of the choices matters, then the number of selections is

$$\underbrace{n \times (n-1) \times (n-2) \times \cdots}_{k \text{ numbers}}$$

But if the order in which the k are chosen does not matter then each set of k items will occur in that grand total

$$k \times (k-1) \times (k-2) \times \cdots \times 2 \times 1$$

times. This is abbreviated to $k!$ (called *k factorial*). So the number of different selections of k items from n is given by

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 1} = \frac{n!}{k!\,(n-k)!}$$

Of course we have assumed that n and k are positive integers with $k \leqslant n$. With the convention that $0! = 1$ the above formula still makes sense if either k or n is zero. If $k > n$ or $k < 0$ then we take $\binom{n}{k}$ to be 0.

Example *Let k and n be integers with $1 \leqslant k \leqslant n$. Use a selection argument to show that*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1} + \cdots + \binom{k}{k-1} + \binom{k-1}{k-1}.$$

Solution The left-hand expression is simply the number of ways of choosing k numbers from $\{1, 2, \ldots, n\}$. How many of those collections have 1 as the lowest? How many have 2 as the lowest? The solution, which generalises the idea used in the previous examples, is now left to the reader. □

Example *How many x^k do you get in the expansion of $(1 + x)^n$?*

Solution

$$(1 + x)^n = \underbrace{(1 + x)(1 + x)(1 + x)\cdots(1 + x)}_{n \text{ lots}}.$$

We could prove this result by induction on n but instead we'll use a selection argument. Multiplying out $(1 + x)^n$ consists of adding up all the terms obtained by multiplying together one entry from the first bracket (namely either 1 or x) times one item from the second, times one from the third, etc. To get an x^k you must therefore have chosen the x from precisely k of the n brackets. So the total number of x^k in the expansion equals the number of ways of choosing k brackets from the n, namely $\binom{n}{k}$. □

It follows from that example that

$$(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{k}x^k + \cdots + \binom{n}{n}x^n.$$

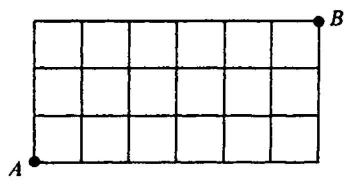
This is the well-known *binomial expansion* and so the numbers $\binom{n}{k}$ are often referred to as the *binomial coefficients*.

Example *Show that*

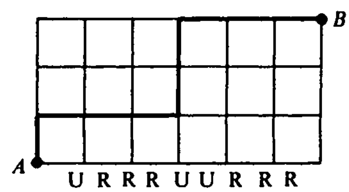
$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

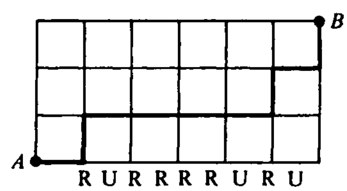
Solution The easiest way to derive this result is to put x equal to 1 in the binomial expansion above. But an alternative method by selection arguments is to count the number of subsets of a set of n items. There are $\binom{n}{0}$ no-element subsets (i.e. just the empty set), $\binom{n}{1}$ one-element subsets, $\binom{n}{2}$ two-element subsets, and so on. Hence the left-hand expression in the example represents the total number of subsets (of all sizes) of a set of n items. Is there a quicker way of counting those subsets? To form a subset you must simply decide for each of the n items whether it is in the subset or not. This is a two-way choice ('in' or 'out') and is made independently for each of the n items. So the total number of ways of choosing a subset is 2^n and the above result follows. □

Example *Imagine that this picture represents a rectangular grid of roads and that you want to walk along the roads from A to B using as short a route as possible. How many different such routes are there? Generalise the result to any size grid.*

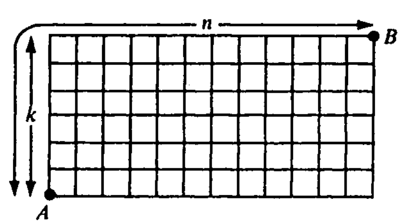


Solution Two typical routes from A to B are illustrated. Any shortest route from A to B must consist of nine moves towards B: in other words each route must consist of nine moves, any three of which must be 'up' (U) and the rest of which must be 'right' (R). For example, the two illustrated routes can be described by the sequences of 'ups' and 'rights' stated. So the number of acceptable routes from A to B equals the number of ways of choosing which three of the nine moves must be 'up'. There are $\binom{9}{3} = 84$ such choices.





A similar argument shows that the number of shortest routes from A to B in the grid illustrated on the right (with n moves necessary, any k of which must be 'up') is $\binom{n}{k}$. □

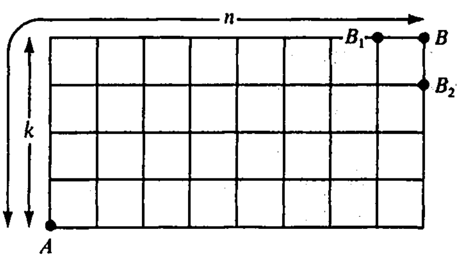


Example *Let k and n be integers with $1 \leqslant k \leqslant n$. Show that*

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Solution There are many ways of proving this identity. The most boring would be to express each side in terms of factorials and to show that you get the same answers. Another way would be to see how many x^k you get in $(1 + x)^n$ and in $(1 + x)(1 + x)^{n-1}$ (which are the same thing). But the solution we give uses the routes in a grid as discussed above.

There are $\binom{n}{k}$ shortest routes from A to B in the grid illustrated. These routes fall into two separate types, those whose final move is 'right' (i.e. those which approach B from B_1) and those whose final move is 'up' (i.e. those which approach B from B_2). Hence



$$\binom{n}{k} = \text{number of routes from } A \text{ to } B$$

$$= (\text{number of routes from } A \text{ to } B_1) + (\text{number of routes from } A \text{ to } B_2)$$

$$= \binom{n-1}{k} + \binom{n-1}{k-1}$$

as required. □

The identity in that example is the basis of *Pascal's triangle* of the binomial coefficients, where each is obtained by adding two earlier ones:

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ & & & & & \searrow & \swarrow & & & & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ & & & & & \vdots & & & & & \end{array}$$

Example *How many solutions are there of the equation*

$$x_1 + x_2 + x_3 + x_4 = 6$$

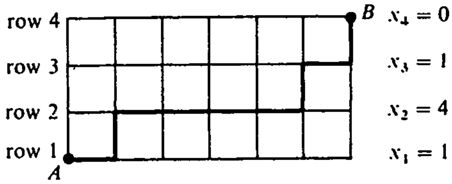
where each x_i is a non-negative integer? How many solutions are there of the equation

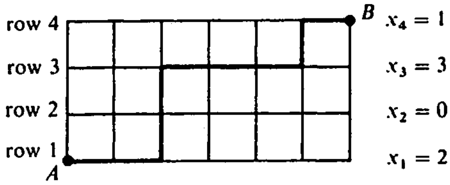
$$x_1 + x_2 + \cdots + x_k = n$$

where each x_i is a non-negative integer?

Solution Again there are many ways of tackling this question (one of which we'll see in the exercises). But as before we choose a solution using routes in a grid. For reasons which will soon become apparent we choose a grid with four horizontal roads and six moves 'right'.

Two shortest routes from A to B are illustrated. If you then regard the number of moves in row i as x_i then these routes translate to the given solutions of the equation $x_1 + x_2 + x_3 + x_4 = 6$.





There is clearly a one-to-one correspondence between the routes and the solutions and it follows that the number of solutions is $\binom{9}{3} = 84$. Similar reasoning shows that the number of solutions of

$$x_1 + x_2 + \cdots + x_k = n$$

equals the number of shortest routes from corner to corner in a grid with k horizontal roads and n moves 'right'. In such a route in this grid there are $n + k - 1$ moves of which any $k - 1$ must be 'up'. Hence the number of solutions is $\binom{n+k-1}{k-1}$. □

We have now considered the *bi*nomial coefficients where, as the prefix 'bi' implies, we are involved with a two-way choice, namely to include or not to include a particular item. We can easily extend this idea to *multi*nomial coefficients.

Example *How many different ten-figure numbers can be formed by writing down the ten digits* 4, 4, 4, 4, 3, 3, 3, 2, 2 *and* 1 *in some order?*

Solution The answer is

$$\frac{10!}{4! \times 3! \times 2!} = 12\,600.$$

To see this we can imitate the solution to our very first example on binomial coefficients. Imagine the long list of 10! numbers formed by the given 10 digits in any order: this list will have many repeats. For example the number

$$4\,231\,442\,343$$

will be in the list $4! \times 3! \times 2!$ times because shuffling around the 4s within their four places, the 3s within their places and the 2s within theirs will not change the number. Similarly each possible number will be in the list $4! \times 3! \times 2!$ times, and the result follows.

Alternatively note that the problem is equivalent to choosing four positions from the ten available for the 4s, then three positions from the remaining six for the 3s, and two positions from the remaining three for the 2s: the position of the 1 is then determined. The number of ways of doing this is

$$\binom{10}{4}\binom{6}{3}\binom{3}{2}\binom{1}{1} = 12\,600$$

as before. □

That answer,

$$\frac{10!}{4!\,3!\,2!\,1!} = \binom{10}{4}\binom{6}{3}\binom{3}{2}\binom{1}{1},$$

can also be thought of as the number of ways of placing ten items into four boxes with four in the first box, three in the second, two in the third, and one in the fourth. In general, if $r \geqslant 2$ and $k_1, k_2, \ldots, k_r$ are integers with $k_1 + k_2 + \cdots + k_r = n$, then the number of ways of placing n items in r boxes with k_1 items in the first box, k_2 in the second, . . . and k_r in the rth is called a *multinomial coefficient* and is denoted by

$$\binom{n}{k_1 \quad k_2 \quad \cdots \quad k_r}.$$

If any of the integers k_i is negative then the coefficient is zero, but if all the k_i are non-negative then either of the arguments used in the last solution leads to

$$\binom{n}{k_1 \quad k_2 \quad \cdots \quad k_r} = \frac{n!}{k_1!\,k_2!\cdots k_r!}.$$

We only use multinomial coefficients in the cases when $r \geqslant 2$ and when the r numbers in the bottom row add to the number in the top row. In particular, notice that when $r = 2$ the multinomial coefficients reduce to the usual binomial coefficient since

$$\binom{n}{k_1 \quad k_2} = \frac{n!}{k_1!\,k_2!} = \frac{n!}{k_1!\,(n-k_1)!} = \binom{n}{k_1}.$$

Also some of the properties of the binomial coefficients easily extend to the multinomial coefficients.

Example *Show that*

$$\binom{n}{k_1 \quad k_2 \quad k_3 \quad \cdots \quad k_r} = \binom{n-1}{k_1 - 1 \quad k_2 \quad k_3 \quad \cdots \quad k_r} + \binom{n-1}{k_1 \quad k_2 - 1 \quad k_3 \quad \cdots \quad k_r} + \cdots + \binom{n-1}{k_1 \quad k_2 \quad k_3 \quad \cdots \quad k_r - 1}$$

Solution We have to place n items in r boxes with k_1 items in the first box, k_2 in the second, ... and k_r in the rth. If the first item is placed in box i then the remaining $n-1$ items must be placed in the r boxes with k_1 items in the first box, k_2 in the second, ..., $k_i - 1$ in the ith, ... and k_r in the rth. Since i can take any value from 1 to r the stated result follows easily. □

Example *The binomial expansion can be written in the form*

$$(a+b)^n = \binom{n}{n \quad 0}a^n + \binom{n}{n-1 \quad 1}a^{n-1}b + \binom{n}{n-2 \quad 2}a^{n-2}b^2 + \cdots + \binom{n}{0 \quad n}b^n$$

$$= \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 = n}} \binom{n}{k_1 \quad k_2} a^{k_1} b^{k_2}.$$

Show that in general

$$(a_1 + a_2 + \cdots + a_r)^n = \sum_{\substack{k_1, \ldots, k_r \geq 0 \\ k_1 + \cdots + k_r = n}} \binom{n}{k_1 \quad \cdots \quad k_r} a_1^{k_1} a_2^{k_2} \cdots a_r^{k_r}.$$

Solution Again we can imitate the method used for the binomial expansion:

$$(a_1 + a_2 + \cdots + a_r)^n = \underbrace{(a_1 + a_2 + \cdots + a_r)(a_1 + a_2 + \cdots + a_r)\cdots(a_1 + a_2 + \cdots + a_r)}_{n \text{ lots}}.$$

When multiplying out these brackets the coefficient of $a_1^{k_1}a_2^{k_2}\cdots a_r^{k_r}$ will equal the number of ways of choosing a_1 from k_1 of the brackets, choosing a_2 from k_2 of the remaining brackets, etc., and the result follows. □

We conclude this introductory chapter with a large selection of exercises: for those marked [H] a helpful (?) hint will be found in the section which starts on page 224 and for those marked [A], which require a numerical answer, the actual answer is given in the section which begins on page 249.

Exercises

1. Show by various methods that

$$\binom{n}{k} = \binom{n}{n-k}. \qquad \text{[H]}$$

2. There are n people in a queue for the cinema (and, being in England, the order of people in the queue never changes). They are let into the cinema in k batches, each batch consisting of one or more persons. In how many ways can the k batches be chosen? [H,A]

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3. How many solutions are there of the equation

$$x_1 + x_2 + \cdots + x_k = n$$

where each x_i is a positive integer? [H,A]

4. (i) Use the answer to exercise 3 to find an alternative verification of the fact that the number of solutions of

$$x_1 + x_2 + \cdots + x_k = n$$

with each x_i a non-negative integer is $\binom{n+k-1}{k-1}$. [H]

(ii) By considering the number of x_i which are zero in (i) and by using the answer to exercise 3 for the positive x_i, show that

$$\binom{n+k-1}{k-1} = \binom{k}{0}\binom{n-1}{k-1} + \binom{k}{1}\binom{n-1}{k-2} + \binom{k}{2}\binom{n-1}{k-3} + \cdots + \binom{k}{k-1}\binom{n-1}{0}.$$

5. In a row of n seats in the doctor's waiting-room k patients sit down in a particular order from left to right. They sit so that no two of them are in adjacent seats. In how many different ways could a suitable set of k seats be chosen? [H,A]

6. By considering colouring k out of n items using one of two given colours for each item, show that

$$\binom{n}{0}\binom{n}{k} + \binom{n}{1}\binom{n-1}{k-1} + \binom{n}{2}\binom{n-2}{k-2} + \cdots + \binom{n}{k}\binom{n-k}{0} = 2^k\binom{n}{k}.$$

7. Show by each of the following methods that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

(i) Use the expansion of $(1 + x)^{2n}$. [H]
(ii) Consider the choice of n people from a set of $2n$ which consists of n men and n women. [H]
(iii) Count the number of routes in a suitable grid. [H]

8. (i) There is a group of $2n$ people consisting of n men and n women from whom I wish to choose a subset, the only restriction being that the number of men chosen equals the number of women chosen. Use the result of exercise 7 to show that the subset may be chosen in $\binom{2n}{n}$ ways.

(ii) I now wish to choose the subset as in (i) and then to choose a leader from the men in the subset and a leader from the women in the subset. Calculate the number of ways of choosing the subset first and then the leaders, and also calculate the number of ways of choosing the leaders first and then the remainder