0. THE INEQUALITIES OF IT ALL

1. Triangle Inequality: $\|x+y\| \le \|x\| + \|y\|$ (I.1c)
2. Cauchy-Schwarz-Bunyakovsky: $|(x,y)| \le (x,x)^{1/2}(y,y)^{1/2}$ (I.4)
3. Bessel: (u_i) orthonormal, $x \in H$, $n \ge 1$, $\sum_{i=1}^{n} |(x,u_i)|^2 \le \|x\|^2$ (II.2)
4. Riesz-Fischer: (two inequalities actually)

   (u_n) complete orthonormal, $\sum_{n=1}^{\infty} a_n u_n$ converges (in H) if and only if

$$\sum_{n=1}^{\infty} |a_n|^2 < +\infty \qquad \text{(II.8)}$$

5. Parseval: (equality), (y_n) complete orthonormal, $x \in H$

$$\|x\|^2 = \sum_{n=1}^{\infty} |(x,y_n)|^2 \qquad \text{(II.14)}$$

6. Holder: $\frac{1}{p} + \frac{1}{q} = 1$, $p \ge 1$ $(q = \infty \text{ if } p = 1)$

$$\begin{aligned} &\sum_{1}^{n} |a_i b_i| \le \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} \left[\sum_{i=1}^{n} |b_i|^q\right]^{1/q} \\ &\left[\sum_{i=1}^{n} |a_i|\right] \max\{|b_i| : 1 \le i \le n\} \text{ if } q = \infty \end{aligned} \qquad \text{(IX.2)}$$

7. Minkowski: $p \ge 1$,

$$\left[\sum_{i=1}^{n} |a_i + b_i|^p\right]^{1/p} \le \left[\sum_{i=1}^{n} |a_i|^p\right]^{1/p} + \left[\sum_{i=1}^{n} |b_i|^p\right]^{1/p} \qquad \text{(IX.3)}$$

8. (weak) Weyl: $T \in K(H)$, $p \ge 1$

$$\sum_{n=1}^{\infty} |\lambda_n(T)|^p \le \sum_{n=1}^{\infty} \sigma_n(T)^p \qquad \text{(IX.8)}$$

9.* Hadamard: (a_{ij}) an $n \times n$ matrix with values in $\mathbb{C}$

$$|\det(a_{ij})| \le \prod_{j=1}^{n} \left(\sum_{i=1}^{n} |a_{ij}|^2\right)^{1/2} \qquad \text{(Appendix B.1)}$$

10.* Weyl: for $T \in K(H)$, $|\lambda_1(T)| \ge |\lambda_2(T)| \ge \cdots$

$$\prod_{i=1}^{n} |\lambda_i(T)| \leq \prod_{i=1}^{n} \sigma_i(T) \qquad \text{(Appendix B.3)}$$

11. Hardy: $p > 1$, $a_i \geq 0$, $A_n = \sum_{i=1}^{n} a_i$,

$$\sum_{n=1}^{m} \left(\frac{A_n}{n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{m} a_n^p \qquad \text{(XI.3)}$$

12.* Comparison of geometric-arithemetic means: $a_i \geq 0$

$$\left(\prod_{i=1}^{n} a_i\right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} a_i \qquad \text{(Appendix C)}$$

(*) Appendices A, B and C can be omitted without any change to the remaining sections. The inequalities marked with * are not used elsewhere in the text. This, in no way, is meant to belittle their importance.

I. PRELIMINARIES

Exercise 1. Read the introduction!

The results we discuss in this monograph concern spectral properties of linear transformations on a very special space: *Hilbert Space.* This opening sentence should be meaningful before too many pages are turned!

The underlying space is a linear (= vector) space. Many of the results to follow are true for real or complex linear spaces but, to make the opening sentence meaningful we will usually assume throughout that the scalar field for our linear spaces is $\mathbb{C}$, the complex numbers.

Thus, let X be a linear space over $\mathbb{C}$. In this space we need a way to measure distance between elements.

I.1 Definition. A *norm*, $\|\bullet\|$, is a function from X into the non-negative reals $\mathbb{R}^+$ satisfying

(a) $\|x\| = 0$ if and only if $x = 0$; (the first "0" is the real number 0 and the second is the zero of the vector space X!)

(b) for each $x \in X$ and each scalar $\alpha \in \mathbb{C}$

$$\|\alpha x\| = |\alpha|\|x\|; \text{ and,}$$

(c) "the triangle inequality" $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

We emphasize that, by definition, $\|x\| \geq 0$ for all $x \in X$.

Let's consider a few norms on *real* spaces to see that the concept of norm generalizes the notion of absolute value, and more generally the notion of the length of a vector.

Some Examples:

(a) Let $\mathbb{R}_1$ denote the real line with the usual arithmetic. The usual absolute value, $|x|$, is a norm.

(b) Consider $\mathbb{R}_n$, the real space of n-tuples $x = (x_1, x_2, \ldots, x_n)$. Let

$$\|x\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}.$$

Then $\|\bullet\|$ is a norm. (We'll have plenty to say about this later).

(c) Define $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ on $\mathbb{R}^n$. Then $\| \bullet \|_1$ is a norm.

(d) Define $\|x\|_\infty = \max(|x_1|, \ldots, |x_n|)$ on $\mathbb{R}_n$. Then $\| \bullet \|_\infty$ is a norm.

The student should check these statements. Similarly, we can define norms on $\mathbb{C}$ and $\mathbb{C}_n$, the space of complex n-tuples. Let us look at a less familiar space.

(e) Let $C[a, b]$ denote the space of real (or complex) valued continuous functions on the interval $[a, b]$. For $f \in C[a, b]$ define $\|f\| = \sup\{|f(t)| : t \in [a, b]\}$. Then $\| \bullet \|$ is a norm on $C[a, b]$. (Here you must use the fact that a continuous real-valued function on $[a, b]$ achieves its maximum).

If one has a linear space X and a norm $\| \bullet \|$ defined on X, define the distance between $x, y \in X$ by $\|x - y\|$. Once we have this distance function given by a norm, we can extend familiar concepts from the calculus to this more general setting.

I.2 Definition. Let (x_n) be a sequence in X and $\| \bullet \|$ a norm on X. We say that (x_n) is a *Cauchy sequence* (with respect to the given norm) if for every $\varepsilon > 0$ there is an integer N such that $m, n \geq N$ implies $\|x_n - x_m\| < \varepsilon$. We say that the sequence (x_n) has a *limit* x in X (or that (x_n) *converges* to x) in the given norm provided that for every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies $\|x_n - x\| < \varepsilon$. We write $\lim_{n\to\infty} \|x_n - x\| = 0$ or $\lim_{n\to\infty} x_n = x$. A function defined on X is *continuous* at $x \in X$ provided that for every sequence (x_n) in X converging to x, the sequence $\big(f(x_n)\big)$ converges to $f(x)$. Here, for convenience, the range of f is assumed to be in a vector space equipped with a norm (although this really isn't essential nor all together desirable).

If "+" and "•" are the operations of addition and scalar multiplication on X, these operations are continuous in the following sense: if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ then $\lim_{n\to\infty} x_n + y_n = x + y$. If $\lim_{n\to\infty} \alpha_n = \alpha$ (in $\mathbb{C}$) then $\lim_{n\to\infty} \alpha_n x_n = \alpha x$. These facts result from the triangle inequality and should be proved by the student.

A linear space X equipped with a norm $\| \bullet \|$ and distance between vectors measured by this norm is, not surprisingly, called a *normed linear space*. If every Cauchy sequence in a normed linear space X has a limit in X then X is said to be *complete*. A complete normed linear space is called a *Banach* space after the great Polish mathematician Stefan Banach. The Hilbert spaces we are after are Banach spaces whose norms have very special properties.

I.3 Definition. Let X be a linear space over $\mathbb{C}$. An *inner -product*, $(\bullet,\bullet)$, is a function defined on $X\times X$, the set of all pairs of elements in X, satisfying

(a) $(x,x)\geq 0$;
(b) $(x,x)=0$ if and only if $x=0$;
(c) $(\alpha x,y)=\alpha(x,y)$ for all $x,y\in X$ and $\alpha\in\mathbb{C}$;
(d) $(x,\alpha y)=\overline{\alpha}(x,y)$ for all $x,y\in X$, $\alpha\in\mathbb{C}$. Here $\overline{\alpha}$ denotes the complex conjugate of α; we will call this property *conjugate homogeneity*.
(e) $(x,y)=(\overline{y,x})$; and
(f) $(x_1+x_2,y)=(x_1+y)+(x_2,y)$ for all x_1, x_2, $y\in X$.

For example in $\mathbb{C}_n$, $(x,y)=\sum_{i=1}^{n}x_i\overline{y}_i$ is an inner-product. Of course, the student should check this!

One of the most important inequalities in mathematics concerns inner-products.

I.4 Theorem. (Cauchy-Schwarz-Bunyakovsky Inequality) If $(\bullet,\bullet)$ is an inner-product on a linear space X then for all $x,y\in X$

$$|(x,y)|\leq(x,x)^{1/2}(y,y)^{1/2}.$$

Proof: For $x,y\in X$ let $b=\frac{(x,y)}{|(x,y)|}$ if $(x,y)\neq 0$ and $b=1$ if $(x,y)=0$; also, let a be an arbitrary real number. Using the properties of an inner-product, we obtain

$$(ax+by,ax+by)=a^2(x,x)+a\overline{b}(x,y)+ba(y,x)+b\overline{b}(y,y)\geq 0.$$

Now, using the definition of b and $(x,y)=(\overline{y,x})$, we obtain

$$(*)\qquad a^2(x,x)+2a|(x,y)|+(y,y)\geq 0.$$

If $x=0$ there is nothing to prove and if $x\neq 0$ $(*)$ has a minimum at $a=\frac{-|(x,y)|}{(x,x)}$ (just take the derivative as a function of a). Putting this value of a into (*) yields the inequality.

One of our main applications of the Cauchy-Schwarz-Bunyakovsky inequality is the following:

I.5 Theorem. If $(\bullet, \bullet)$ is an inner-product on $X \times X$ then $(x,x)^{1/2}$ is a norm on X.

Proof: Everything in the definition of a norm follows from the corresponding properties of the inner-product except the triangle inequality. For $x, y \in X$ (writing $\|x\| = (x,x)^{1/2}$) we have

$$\begin{aligned} \|(x+y)\|^2 &= (x,x) + (x,y) + (y,x) + (y,y) \\ &= \|x\|^2 + 2\mathrm{Re}(x,y) + \|y\|^2 \\ &\leq \|x\|^2 + 2|(x,y)| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2. \end{aligned}$$

The last inequality is just the Cauchy-Schwarz-Bunyakovsky inequality. The symbol Rez denotes the real part of $z \in \mathbb{C}$.

You should now check that on $\mathbb{R}_n$ or $\mathbb{C}_n$

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

is a *norm* coming from the *inner-product*

$$(x,y) = \sum_{i=1}^{n} x_i \overline{y}_i.$$

A Banach space whose norm $\| \bullet \|$ comes from an inner-product via $\|x\| = (x,x)^{1/2}$ is called a *Hilbert-space* in honor of the great German mathematician David Hilbert.

Exercise 2. If $\|\bullet\|$ is such a norm, prove the parallelogram law: $\|x+y\|^2 = 2\left[\|x\|^2 + \|y\|^2\right]$.

A normed linear space (not complete) is called a *pre-Hilbert space* if its norm comes from an inner-product in this fashion. Hilbert and pre-Hilbert spaces are called inner-product spaces.

We can now present the main example of Hilbert spaces (at least for our purposes!).

I.6 Example. The space ℓ_2.

Let $\ell_2 = \left\{(a_n) \middle| \sum_{n=1}^{\infty} |a_n|^2 < +\infty\right\}$ with coordinate-wise arithemetic. Here the (a_n) are sequences of complex numbers (although at this stage they could be real sequences). That ℓ_2 is a linear space follows from $ab \leq \frac{1}{2}\left[a^2 + b^2\right]$. If $a = (a_n)$, $b = (b_n) \in \ell_2$ then $(a, b) = \sum_{n=1}^{\infty} a_n \bar{b}_n$ is an inner product. Indeed $(|a_n|)$ and $(|b_n|)$ are in ℓ_2 and for fixed N the Cauchy-Schwarz-Bunyakovsky inequality gives

$$\begin{aligned}\sum_{n=1}^{N} |a_n||\bar{b}_n| &\leq \left(\sum_{n=1}^{N} |a_n|^2\right)^{1/2} \left(\sum_{n=1}^{N} |b_n|^2\right)^{1/2} \\ &\leq \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} |b_n|^2\right)^{1/2}.\end{aligned}$$

Letting $N \to \infty$ yields

$$\sum_{n=1}^{\infty} |a_n||b_n| \leq \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} |b_n|^2\right)^{1/2}$$

so (a, b) is well-defined. It is routine to check that (a, b) has all the properties of an inner-product. In particular observe that

$$\left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2} = (a, a)^{1/2}$$

is the inner-product norm on ℓ_2.

With this norm ℓ_2 is complete! This fact is non-trivial but a proof should be attempted by the student (perhaps with a lot of help from the instructor). Thus ℓ_2 is a Hilbert space. Let e_n denote the sequence with 1 the n$^{\text{th}}$ term and the remaining terms 0. Clearly each $e_n \in \ell_2$ and any finite set $\{e_1, e_2, \ldots, e_N\}$ is linearly independent, so ℓ_2 is *not* a finite-dimensional linear space!

From now on H will denote a Hilbert space. We should remark that the inner-product $(\bullet, \bullet)$ on $H \times H$ is continuous in the following sense: If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ then $\lim_{n\to\infty} (x_n, y_n) = (x, y)$. Clearly,

the statement $\lim_{n\to\infty}(x_n, y_n) = (x, y)$ is equivalent to $\lim_{n\to\infty}[(x_n - x, y_n) + (x, y_n - y)] = 0$ (why?) and this is immediate from the Cauchy-Schwarz-Bunyakovsky inequality. (You need the fact that a Cauchy sequence is bounded:
$\sup_n ||y_n|| \leq M$ for some $M > 0$.) We will now begin studying the special properties enjoyed *only* by Hilbert spaces among the Banach spaces.

Remarks, Exercises and Hints

The axioms for finite dimensional Euclidean Spaces (our spaces $\mathbb{R}_n(\mathbb{C}_n)$) were given by Herman Weyl about 1910. Hilbert spaces were studied by David Hilbert and his Tübingen school but not in an abstract fashion. The axioms for what we now call Hilbert space were given by John VonNeumann in 1927. In 1932 Stefan Banach introduced the axioms for what he called "spaces of type B". Mathematicians took the hint and thus Banach spaces came into being. Earlier, similiar ideas had been used by Norbert Wiener and T. H. Hildebrandt. In the older literature you will find Hildebrandt never yielded and referred to Banach spaces as NLCS(normed, linear complete spaces)!

The notion of inner-product (or dot product or scalar product) goes back to the early days of vector analysis.

The Cauchy-Schwarz-Bunyakovsky inequality was proved for finite sums by Cauchy (1821), for integrals by Bunyakovsky (1859) and rediscovered by Schwarz in 1885. We will see that the Cauchy-Schwarz-Bunyakovsky inequality is a special case of Hölders inequality.

1. In $\mathbb{R}_2$, view (a,b) as the directed segment (vector) from (0,0) to (a,b). Given two vectors u and v let Θ be the angle between them satisfying $0 \leq \Theta \leq \pi$.

   a. Prove that $(u,v) = ||u|| ||v|| \cos\Theta$ and conclude that Θ is acute (obtuse) if $(u,v) > 0\big((u,v) < 0\big)$ and $\Theta = \frac{\pi}{2}$ if and only if $(u,v) = 0$.

2. In definition I.3, (d) is superfulous.

3. Show that an inner product is additive in the second variable:

$$(x, y_1 + y_2) = (x, y_1) + (x, y_2) \text{ for all } x, y_1, y_2.$$

4. If X is an inner product space and $(x,u) = (x,v)$ for all $x \in X$ then $u = v$.

5. If X is a real inner product space, show that $(x-y, x+y) = 0$ if and only if $||x|| = ||y||$.

6. If X is an inner product space and $\lim_{n\to\infty}(x_n, x) = (x,x)$ and $\lim\limits_{n\to\infty} ||x_n|| = ||x||$ then $\lim_{n\to\infty} x_n = x$.

7. Do the norms $||x||_1 and ||x||_\infty$ defined on $\mathbb{R}_n, n > 2$ come from an inner-product? What about $n = 1$?

8. Answer problem 7 for the space $C[a, b]$ with the supremum norm.

Finally a more difficult problem.

In Exercise 2 of this chapter you were asked to prove the parallelgram law from inner-product spaces.

9. (Jordan, VonNeumann) Prove that if X is a normed linear space with a norm satisfying (*) $||x + y||^2 + ||x - y||^2 = 2\left[||x||^2 + ||y||^2\right]$ the norm comes from an inner-product.

   [HINT] If X is real, define

$$(x, y) = \frac{1}{4}\left[||x + y||^2 - ||x - y||^2\right];$$

if X is complex, define

$$\text{Re}(x, y) = \frac{1}{4}\left[||x+y||^2 - ||x-y||^2\right] \text{ and } Im(x, y) = \frac{1}{4}\left[||x+iy||^2 - ||x-iy||\right]$$

(Rez and Imz denote the real and imaginary parts of the complex number z). To prove $(+)(\alpha x, y) = \alpha(x, y)$ use induction to prove $(+)$ for $\alpha = \frac{1}{n}$. Then, use induction again to prove $(+)$ for $\frac{m}{n}$. If α is irrational write $\alpha = \lim_{n\to\infty} r_n$ with r_n rational and invoke a continuity argument. For the complex case use the above ideas applied to the real and imaginary parts of (x, y).

For the remainder of the proof use the following big hint of M.M. Day:
In the parallelogram equality replace x by $x \pm z$ obtaining

$$\begin{aligned}||x + z + y||^2 + ||x + z - y||^2 &= ||x - z + y||^2 - ||x - z - y||^2 \\ &= 2\left[||x + z||^2 + ||y||^2 - ||x - z||^2 - ||y||^2\right].\end{aligned}$$

Use this to show that

$$4(x + y, z) + 4(x - y, z) = 8(x, z).$$

Letting $x + y = a \qquad x - y = b$ yields

$$(a, z) + (b, z) = (a + b, z).$$