Symmetries of modular surfaces

M. Akbas and D. Singerman

To Murray Macbeath on the occasion of his retirement

1. Introduction

Let $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ be the rational modular group and $\Gamma(N)$, $\Gamma_0(N)$ denote the subgroups represented by the matrices

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\},$$

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \,\middle|\, c \equiv 0 \bmod N \right\}$$

respectively. Let U denote the upper half-plane and $U^* = U \cup \mathbb{Q} \cup \{\infty\}$. Let $X(N) = U^*/\Gamma(N)$, $X_0(N) = U^*/\Gamma_0(N)$. Then $X(N)$, $X_0(N)$ are compact Riemann surfaces.

A Riemann surface X is symmetric if it admits an anticonformal involution (or as we shall call it, *symmetry*) t. If X has genus g then by Harnack's Theorem, the fixed point set of t consists of k simple closed curves where $0 \leq k \leq g+1$. We shall call each such simple closed curve a *mirror* of t. The mirrors of a symmetry may either (i) divide the surface X into two homeomorphic components or (ii) not divide X. In case (i) we say that t has species $+k$ and in case (ii) we say that t has species $-k$. It follows that the species of t has a $+$ (respectively $-$) sign if and only if $X/\langle t\rangle$ is orientable (respectively non-orientable). See [3] for an account of the general theory.

As the transformation $z \to -\bar{z}$ normalizes $\Gamma(N)$ and $\Gamma_0(N)$ it induces a symmetry on the surfaces $X(N)$ and $X_0(N)$ (which by abuse of language we call "the symmetry $z \to -\bar{z}$ of $X(N)$, or $X_0(N)$"). The surfaces $X_0(N)$ also admit a symmetry induced by the "Fricke symmetry" $z \to 1/N\bar{z}$ and we also discuss this.

Our aim here is to summarize the work done that enables us to find the species of these symmetries. As the surfaces $X(N)$, $X_0(N)$ also have cusps (at the projections of the parabolic fixed points) another question of interest is to determine the number of cusps on each mirror. The work we report on comes mainly from four sources. The first two ([7], [10]) were concerned with real points on modular curves and are probably not well-known to workers on discrete groups and Riemann surfaces. The second two are from the Ph.D. theses of the first author and Stephen Harding ([1], [4]). Also see [11] for a number-theoretic application of similar ideas.

These questions are closely related to that of determining the signatures of the NEC groups

$$\widehat{\Gamma}(N) = \langle \Gamma(N), z \to -\bar{z} \rangle, \qquad \widehat{\Gamma}_0(N) = \langle \Gamma_0(N), z \to -\bar{z} \rangle,$$

$$\Gamma_F(N) = \langle \Gamma_0(N), z \to 1/N\bar{z} \rangle.$$

We recall that the signature of a NEC group Λ has the form

$$(g;\ \pm;\ [m_1, \ldots, m_r];\ \{(n_{11}, \ldots, n_{1s_1}) \ldots (n_{k1}, \ldots, n_{ks_k})\})$$

where g is the genus of U/Λ, $m_1, \ldots, m_r$ are the periods of Λ and n_{ij} are the link periods. (See [2], [3], [9], [12]). In these references the groups Λ have compact quotient space and so the integers m_i, n_{ij} are finite. In this paper the NEC groups are all commensurable with the modular group and so have parabolic elements. As usual these are represented by infinite periods so that the number of infinite periods in a period cycle correspond to the number of cusps around the hole represented by that period cycle.

EXAMPLE. The extended modular group

$$\widehat{\Gamma} = \widehat{\Gamma}(1) = \widehat{\Gamma}_0(1) = \langle \Gamma, z \to -\bar{z} \rangle \cong \mathrm{PGL}(2, \mathbb{Z}).$$

This is generated by 3 reflections: $c_1 : z \to -\bar{z}$, $c_2 : z \to 1/\bar{z}$, $c_3 : z \to -\bar{z}-1$ or in terms of matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

the fundamental domain being bounded by the 3 axes of reflection

$$\mathrm{Re}(z) = 0, \qquad |z| = 1, \qquad \mathrm{Re}(z) = \frac{1}{2}.$$

Its boundary contains elliptic fixed points at i and $(1 + i\sqrt{3})/2$ of orders 2 and 3 and there is a parabolic fixed point at ∞. These correspond to the elements c_1c_2 of order 2, c_2c_3 of order 3 and the parabolic c_1c_3. The signature of $\widehat{\Gamma}$ is $\{0; +; [-]; \{(2,3,\infty)\}$ and the quotient U^*/Γ is a disc whose boundary contains two branch points and one cusp.

REMARK. If Λ is a NEC group with sense-reversing transformations then we let Λ^+ denote the subgroup of index 2 consisting of the sense-preserving transformations of Λ. Then U/Λ^+ (or U^*/Λ^+ if appropriate) is the canonical double cover of U/Λ (or U^*/Λ). (See [2], 0.1.12). The mirrors of U/Λ^+ are in one-to-one correspondence with the boundary components of U/Λ. As $(\widehat{\Gamma}(N))^+ = \Gamma(N)$, $(\widehat{\Gamma}_0(N))^+ = \Gamma_0(N)$, the mirrors of $X(N)$, (respectively $X_0(N)$), correspond to the boundary components of $X(N) = U^*/\widehat{\Gamma}(N)$ (respectively $\widehat{X}_0(N) = U^*/\widehat{\Gamma}_0(N)$).

2. The mirrors of X(N)

To describe the number of mirrors of $X(N)$ we introduce an arithmetic function $\alpha(N)$ defined as follows: $\alpha(N)$ is the least positive integer such that $2^{\alpha(N)} \equiv \pm 1 \bmod N$. If U_N denotes the groups of units mod N then $\alpha(N)$ is the order of the image of 2 in $U_N/\{\pm 1\}$ so that if $N > 2$, $2\alpha(N) \big| \phi(N)$, where ϕ is Euler's function. The following Theorems are proved in [1], [7].

THEOREM 1. *The number of mirrors of the symmetry $z \to -\bar{z}$ of $X(N)$ is given by*

$$\begin{cases} \phi(N)/2\alpha(N) & \text{if } N > 1 \text{ is odd} \\ \phi(N)/2 & \text{if } N > 2 \text{ is even} \\ 1 & \text{if } N = 2\ . \end{cases}$$

THEOREM 2. *The number of cusps on each mirror is given by*

$$\begin{cases} 2\alpha(N) & N \text{ odd} \\ 6 & N \equiv 2 \bmod 4,\ N > 2 \\ 4 & N \equiv 0 \bmod 4 \\ 3 & N = 2 \end{cases}$$

The proofs in [1] and [7] are rather different. In [1] the proof is algebraic and uses Hoare's Theorem on subgroups of NEC groups [5]. This theorem gives a general method for computing the signature of a subgroup Δ_1 of a NEC group Δ given the signature of Δ and the permutation representation of the generators of Δ on the right Δ_1-cosets. In our case $\Delta = \widehat{\Gamma}$, $\Delta_1 = \widehat{\Gamma}(N)$ and the generators of $\widehat{\Gamma}$ are $c_1,\ c_2,\ c_3$ above.

In [7], a more geometric approach is used, and involves calculating the cusps on the mirrors. We denote the $\Gamma(N)$-orbit of a rational number a/b by $\begin{pmatrix} a \\ b \end{pmatrix}$. The reflection $z \to -\bar{z}$ of $X(N)$ fixes $\begin{pmatrix} a \\ b \end{pmatrix}$ if $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ b \end{pmatrix}$. Such a fixed cusp is called a *real cusp*. (For an account of the connection between NEC groups and real algebraic geometry see [2].) For example, if N is odd the real cusps on $X(N)$ are, according to lemma 1 of [7], of the form $\left\{ \begin{pmatrix} u \\ N \end{pmatrix}, \begin{pmatrix} N \\ u \end{pmatrix} \,\middle|\, 1 \le u \le \frac{N-1}{2},\ (u, N) = 1 \right\}$. This gives a total of $\phi(N)$ cusps as implied by Theorems 1 and 2. Two real cusps are joined if they have lifts in U^* which are joined by an axis of reflection of $\widehat{\Gamma}(N)$. This axis then projects to a segment of a mirror on $X(N)$. By finding all such segments we can find all the mirrors and the number of cusps on each mirror.

The sign of the species. Theorem 1 gives the number of mirrors of the reflection $z \to -\bar{z}$ of $X(N)$. We now investigate whether the mirrors separate or do not separate the surface, i.e. whether the species has a + sign or − sign. (See §1). This sign is the same as the sign in the signature of $\widehat{\Gamma}(N)$ and this can be determined by Theorem 2 of [6]. We consider the Schreier coset graph $\mathcal{H} = \mathcal{H}(\widehat{\Gamma}, \widehat{\Gamma}(N), \Phi)$ whose vertices are the $\widehat{\Gamma}(N)$-cosets of $\widehat{\Gamma}$ and where $\Phi = \{c_1, c_2, c_3\}$, the generators of $\widehat{\Gamma}$. We then form the graph $\bar{\mathcal{H}}$ by deleting the loops from $\mathcal{H}$. Then $\widehat{\Gamma}(N)$ is orientable if and only if all circuits of $\bar{\mathcal{H}}$ have even length. Harding [4] investigated the orientability in the case $N = p$ a prime. We describe his method. First of all let

$$A = c_1c_2c_3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \qquad \text{Then} \qquad A^k = \begin{pmatrix} u_k & u_{k+1} \\ u_{k+1} & u_{k+2} \end{pmatrix}$$

where u_k is the k^{th} Fibonacci number, $(u_1 = 0, u_2 = 1, u_{k+2} = u_{k+1} + u_k)$. We consider the path that corresponds to the word

$$c_1c_2c_3c_1c_2c_3 \dots c_1c_2c_3 = A^k.$$

To find the corresponding path in $\bar{\mathcal{H}}$ we search for the loops in this path. These occur if either

$$\text{(i) } \widehat{\Gamma}(p)A^k c_1 = \widehat{\Gamma}(p)c_1, \qquad \text{(ii) } \widehat{\Gamma}(p)A^k c_1 c_2 = \widehat{\Gamma}(p)c_1$$
$$\text{or} \qquad \text{(iii) } \widehat{\Gamma}(p)A^{k+1} = \widehat{\Gamma}(p)c_1 c_2.$$

It is easy to calculate that (i) occurs if and only if $u_{k+1} \equiv 0 \bmod p$, (ii) can occur only if $p = 3$ and (iii) never occurs. Thus if $p > 3$, the only loops in $\mathcal{H}$ occur when $u_{k+1} \equiv 0 \bmod p$. Also $u_k u_{k+2} - u_{k+1}^2 = \det A^k = (-1)^k$. If $u_{k+1} \equiv 0 \bmod p$ then $u_{k+2} \equiv u_k \bmod p$ and so $u_k^2 \equiv (-1)^k \bmod p$. Now suppose that $p \equiv 3 \bmod 4$. Then -1 is not a square mod p so that k is even, $u_k \equiv \pm 1 \bmod p$ and so $A^k \in \widehat{\Gamma}(p)$. A loop does occur at the beginning of the path as $c_1 \in \widehat{\Gamma}(p)$. The path closes again when we have reached $(c_1 c_2 c_3)^k$ with k the first integer such that $u_{k+1} \equiv 0 \bmod p$ and there are no other loops before then. Thus the corresponding circuit in $\bar{\mathcal{H}}$ has length $3k - 1$ which is odd as k is even. Thus we have proved (using [6] Theorem 2)

THEOREM 3. *If $p > 3$ and $p \equiv 3 \bmod 4$ is prime then the mirrors of the symmetry $z \to -\bar{z}$ of $X(p)$ do not separate $X(p)$.*

By drawing the Schreier coset graphs for $p = 2, 3, 5$ Harding showed that $\widehat{X}(2), \widehat{X}(3)$ and $\widehat{X}(5)$ are orientable (so the mirrors do separate in these cases). For other primes $p \equiv 1 \bmod 4$ he considered the element

$$B = c_1 c_2 c_3 (c_1 c_3)^{s-1} = \begin{pmatrix} 0 & 1 \\ 1 & s \end{pmatrix} \qquad (s \in \mathbb{N}).$$

Then
$$B^k = \begin{pmatrix} t_k & t_{k+1} \\ t_{k+1} & t_{k+2} \end{pmatrix}$$

with $t_1 = 0$, $t_2 = 1$, $t_{k+2} = s t_{k+1} + t_k$, a generalized Fibonacci sequence. By pursuing an analysis similar to the above Harding showed that $X(p)$ is non-orientable for all primes p with $5 < p < 1000$. Since for non-orientability we only need one circuit in $\mathcal{H}$ of odd length, it seems very likely that $X(p)$ is non-orientable for all primes $p > 5$.

To illustrate the results of this section we give the species of the symmetries $z \to -\bar{z}$ of $X(p)$ for all primes $p < 100$.

p	2	3	5	7	11	13	17	19	23	29	31	37
species	+1	+1	+1	−1	−1	−1	−2	−1	−1	−1	−3	−1

p	41	43	47	53	59	61	67	71	73	79	83	89	97
species	−2	−3	−1	−1	−1	−1	−1	−1	−4	−1	−1	−4	−2

3. The mirrors of $X_0(N)$

The number of mirrors of the symmetry $z \to -\bar{z}$ of $X_0(N)$ and the number of cusps on each mirror were calculated by the first author in [1] using Hoare's Theorem and by Ogg [10] using a technique similar to Jaffee's described previously. The results are as follows. (The notation $m \| n$ means that m is an exact divisor of n, i.e. $m | n$ and $(m, n/m) = 1$, and r is the number of distinct prime factors of N).

THEOREM 4. *The number of mirrors of the symmetry $z \to -\bar{z}$ of $X_0(N)$ and the number of cusps on each mirror is given by the following table*

N	*odd*	2	$2\|N,\ N > 2$	4	$4\|N,\ N > 4$	$8 \mid N$
mirrors	2^{r-1}	*1*	2^{r-2}	*1*	2^{r-2}	2^{r-1}
cusps	*2*	*2*	*4*	*3*	*6*	*4*

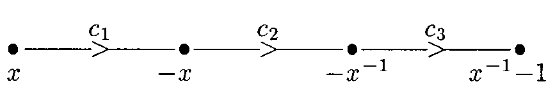
The sign of the species. In [4], Harding calculated the sign of the species of the symmetry $z \to -\bar{z}$ of $X_0(p)$ for all primes p. We describe his method. The group $\widehat{\Gamma} \cong \mathrm{PGL}(2, \mathbb{Z})$ acts transitively on the $p+1$ points of the projective line $GF(p) \cup \{\infty\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : t \to \frac{at+b}{ct+d}.$$

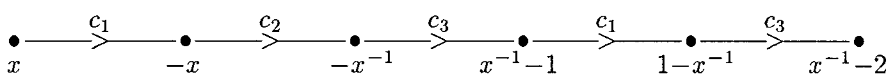
In particular c_1, c_2, c_3 act as follows:

$$c_1 : t \to 1/t, \qquad c_2 : t \to 1/t, \qquad c_3 : t \to -1-t$$

so that c_1 fixes $0, \infty$, c_2 fixes ± 1, c_3 fixes $-\frac{1}{2}, \infty$. The stabilizer of ∞ has index $p+1$ and contains $\Gamma_0(p)$ and c_1 . Hence $\mathrm{Stab}(\infty) = \widehat{\Gamma}_0(p)$. Thus the vertices of the Schreier coset graph $\mathcal{H}_0(p) = \mathcal{H}(\widehat{\Gamma}, \widehat{\Gamma}_0(p), \Phi)$, where $\Phi = \{c_1, c_2, c_3\}$, can be identified with $GF(p) \cup \{\infty\}$. As before we form the graph $\overline{\mathcal{H}}_0(p)$ obtained by deleting the loops of $\mathcal{H}_0(p)$, and if we find circuits of odd length then the sign in the signature of $\Gamma_0(p)$ is $-$. We first consider the word $c_1 c_2 c_3$. If we begin at $x \in GF(p) - \{O\}$ we get the path

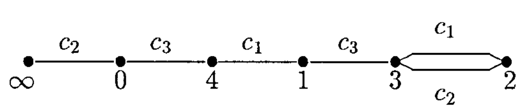


If $x \neq \pm 1$ then c_2 does not fix $-x$. If $x \neq 0$ or 2 then c_3 does not fix $-x^{-1}$. This path closes if $x = x^{-1} - 1$ or $x = (-1 \pm \sqrt{5})/2$. Thus if the Legendre symbol $\left(\frac{5}{p}\right) = +1$ then we can find x so that the above path is a triangle. We cannot have loops unless $p = 2$. For example, $1 = (-1 \pm \sqrt{5})/2$ is not true in $GF(p)$ if $p > 2$. Thus if $\left(\frac{5}{p}\right) = +1$, $(p > 2)$ then $\bar{\mathcal{H}}_0(p)$ has a triangle. We now consider the word $c_1c_2c_3c_1c_3$.



This gives a closed path if $x = x^{-1} - 2$ or $x = -1 \pm \sqrt{2}$. Thus if $\left(\frac{2}{p}\right) = +1$ we can find x giving a closed path. It is now possible to get loops. For example $-x^{-1} = x^{-1} - 1$ gives $x = 2$ but if $p = 7$, $2 \equiv 1 + \sqrt{2}$. However if $\left(\frac{2}{p}\right) = +1$ (equivalent to $p \equiv \pm 1 \bmod 8$) and if $p > 7$ then there are no loops and we have a circuit of odd length in $\bar{\mathcal{H}}_0(p)$.

Thus if $\left(\frac{5}{p}\right) = +1$ or $\left(\frac{2}{p}\right) = +1$ and $p > 7$ then we can find a circuit of odd length. Now we consider the word $c_1c_3c_1c_2c_3(c_1c_3)^4$ of length 13. We find a closed circuit at x if $x = -2 \pm \sqrt{10}$ which exists if $\left(\frac{10}{p}\right) = +1$, and in particular if $\left(\frac{5}{p}\right) = \left(\frac{2}{p}\right) = -1$. We also find that this circuit can only have loops if $p = 3$, 13 or 31. However $\left(\frac{5}{31}\right) = +1$ so this case has been covered already. Hence we can find a circuit of odd length in $\bar{\mathcal{H}}_0(p)$ for all primes p except $p = 2, 3, 5, 7, 13$. If we draw the coset graphs $\bar{\mathcal{H}}_0(p)$ in these cases we see that there are no circuits of odd length. For example if $p = 5$ we get



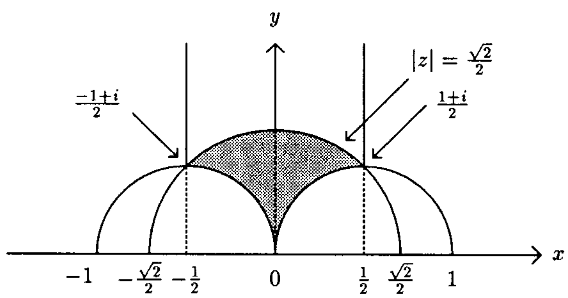
We have therefore proved

THEOREM 5. *If $p = 2, 3, 5, 7, 13$ then the mirrors of the symmetry $z \to -\bar{z}$ of $X_0(p)$ separate the surface. For all other primes the mirrors do not separate.*

Mirrors of the Fricke reflection. The group $\Gamma_0(N)$ is normalized by the Fricke involution $W_N : z \to -1/Nz$ and thus it is also normalized by the reflection $\bar{W}_N : z \to 1/N\bar{z}$ which we shall call the *Fricke reflection*. Therefore $\bar{W}_N$ induces a symmetry $\bar{w}_N$ of $X_0(N)$.

In contrast to the previous cases $\bar{w}_N$ may have mirrors without cusps.

EXAMPLE. $N = 2$. The following diagram shows a fundamental domain for $\Gamma_0(2)$ divided into two by the fixed axis $|z|^2 = \frac{1}{2}$ of $\bar{W}_2$.



There are two cusps at 0 and ∞ and one orbit of elliptic fixed points of period 2 at $(\pm 1 + i)/2$. Thus the mirror of $\bar{w}_2$ has no cusps (and one branch point corresponding to the elliptic fixed points). More generally we have the following simple result.

THEOREM 6. *$\bar{w}_N$ fixes a cusp on a mirror if and only if N is a perfect square.*

PROOF. If $\bar{w}_n$ fixes a cusp then for some $x \in \mathbb{Q} \cup \{\infty\}$ and $T \in \Gamma_0(N)$, $\bar{W}_N : x \to T(x)$. If

$$T = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \qquad (ad - bcN = 1)$$

then $S = \bar{W}_N T$ fixes x. Now S reverses orientation. If it were a glide reflection then S^2 would be a hyperbolic element of the modular group fixing a point of $\mathbb{Q} \cup \{\infty\}$ which is impossible so that S is a reflection. As

$$S = \begin{pmatrix} cN & d \\ aN & bN \end{pmatrix}$$

we conclude that $b + c = 0$. As S fixes x, $aNx^2 + 2bNx - d = 0$. The discriminant is $4b^2N^2 + 4adN = 4N$ which must be a perfect square and the result follows. Conversely, if $N = n^2$ then $\bar{W}_N$ fixes $\frac{1}{n}$. ∎

In [1], the first author obtained the following result (c.f. [10]). First we need some notation. Let A be the number of solutions of $x^2 \equiv -1 \bmod n$. If solutions exist then $A = 2^{\sigma+\tau}$ where σ is the number of odd prime divisors of n and $\tau = 0, 1, 2$ according as $4 \nmid n$, $4 \| n$ or $8 | n$. ([8] p.65). We also let k be the order of 4 in U_N, the group of units mod N.

THEOREM 7. *Let $N = n^2$. Then*

(i) If n is odd then $\bar{w}_n$ has A boundary components containing k cusps and $\dfrac{\phi(n)}{2k} + \dfrac{A}{2}$ boundary components with 2k cusps.

(ii) If n is even then $\bar{w}_n$ has A boundary components with 1 cusp and $\dfrac{\phi(n)}{2} + \dfrac{A}{2}$ boundary components with 2 cusps.

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Karadeniz Üniversitesi
Turkey

University of Southampton
Southampton, England

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Lifting group actions to covering spaces

M. A. Armstrong

To Murray Macbeath on the occasion of his retirement

Several authors (Bredon [2]; Conner and Raymond [3]; Gottlieb [4]; Rhodes [5]) have considered the following question. Given an action of a topological group G on a space X, together with a covering space $\widetilde{X}_H$ of X, when does this action lift to an action of G on $\widetilde{X}_H$? We propose a systematic approach which unifies and extends previous results. In particular we avoid unnecessary local restrictions on G and X, and we verify the *continuity* of the lifted actions.

Our first task is to fix some notation and terminology. Let X be a path connected, locally path connected space with a chosen base point p, and let $\widetilde{X}_H$ denote the covering space of X which corresponds to the subgroup H of $\pi_1(X,p)$. We shall assume throughout that G is a topological group which acts in a continuous fashion as a group of homeomorphisms of X.

Suppose G also acts on a space Z, and that $f : Z \to X$ is an equivariant map which sends the point q of Z to p. We say that H is $(f,G)-$*invariant* providing the homotopy class

$$\langle f(\gamma).g(\alpha).f(\gamma^{-1})\rangle$$

belongs to H for every group element g in G, loop α based at p in X, and path γ which joins q to $g(q)$ in Z. (When $Z = X$ and f is the identity map, we have a "G-invariant subgroup" in the sense of [1].)

