

## Contents

Introduction	ix
1 Algebraic varieties: definition and existence	1
1.1 Spaces with functions	1
1.2 Varieties	2
1.3 The existence of affine varieties	4
1.4 The nullstellensatz	5
1.5 The rest of the proof of existence of affine varieties / subvarieties	8
1.6 $\mathbf{A}^n$ and $\mathbf{P}^n$	10
1.7 Determinantal varieties	11
2 The preparation lemma and some consequences	13
2.1 The lemma	13
2.2 The Hilbert basis theorem	15
2.3 Irreducible components	16
2.4 Affine and finite morphisms	18
2.5 Dimension	20
2.6 Hypersurfaces and the principal ideal theorem	21
3 Products; separated and complete varieties	25
3.1 Products	25

vi	<i>Contents</i>	
3.2	Products of projective varieties	27
3.3	Graphs of morphisms and separatedness	28
3.4	Algebraic groups	30
3.5	Cones and projective varieties	31
3.6	A little more dimension theory	32
3.7	Complete varieties	33
3.8	Chow's lemma	34
3.9	The group law on an elliptic curve	35
3.10	Blown up $\mathbb{A}^n$ at the origin	36
4	Sheaves	38
4.1	The definition of presheaves and sheaves	38
4.2	The construction of sheaves	42
4.3	Abelian sheaves and flabby sheaves	46
4.4	Direct limits of sheaves	50
5	Sheaves in algebraic geometry	54
5.1	Sheaves of rings and modules	54
5.2	Quasi-coherent sheaves on affine varieties	56
5.3	Coherent sheaves	58
5.4	Quasi-coherent sheaves on projective varieties	61
5.5	Invertible sheaves	62
5.6	Operations on sheaves that change spaces	65
5.7	Morphisms to projective space and affine morphisms	68
6	Smooth varieties and morphisms	70
6.1	The Zariski cotangent space and smoothness	70
6.2	Tangent cones	72
6.3	The sheaf of differentials	75
6.4	Morphisms	80
6.5	The construction of affine morphisms and normalization	82
6.6	Bertini's theorem	83
7	Curves	85
7.1	Introduction to curves	85
7.2	Valuation criteria	87
7.3	The construction of all smooth curves	88
7.4	Coherent sheaves on smooth curves	90
7.5	Morphisms between smooth complete curves	92

<i>Contents</i>		vii
7.6	Special morphisms between curves	94
7.7	Principal parts and the Cousin problem	96
8	Cohomology and the Riemann–Roch theorem	98
8.1	The definition of cohomology	98
8.2	Cohomology of affines	100
8.3	Higher direct images	102
8.4	Beginning the study of the cohomology of curves	104
8.5	The Riemann–Roch theorem	106
8.6	First applications of the Riemann–Roch theorem	108
8.7	Residues and the trace homomorphism	110
9	General cohomology	113
9.1	The cohomology of $\mathbf{A}^n - \{0\}$ and $\mathbf{P}^n$	113
9.2	Čech cohomology and the Künneth formula	114
9.3	Cohomology of projective varieties	116
9.4	The direct images of flat sheaves	118
9.5	Families of cohomology groups	120
10	Applications	124
10.1	Embedding in projective space	124
10.2	Cohomological characterization of affine varieties	125
10.3	Computing the genus of a plane curve and Bezout’s theorem	126
10.4	Elliptic curves	128
10.5	Locally free coherent sheaves on $\mathbf{P}^1$	129
10.6	Regularity in codimension one	130
10.7	One dimensional algebraic groups	131
10.8	Correspondences	132
10.9	The Riemann–Roch theorem for surfaces	139
	Appendix	139
A.1	Localization	141
A.2	Direct limits	143
A.3	Eigenvectors	144
	Bibliography	146
	Glossary of notation	149
	Index	155