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Algebraic varieties: definition and existence

In this chapter we meet the category of algebraic varieties. We will give their definitions and discuss the subsequent question of their existence. We begin the discussion with the larger category of spaces with functions.

1.1 Spaces with functions

Let k be a fixed field. *A space with functions* is a topological space X together with the assignment to each open subset U of X of a k-algebra $k[U]$ of k-valued functions on U, which we say consists of all *regular* functions on U, satisfying the properties (a) and (b) below.
The conditions are:

(a) Let U be the union $\bigcup U_\alpha$ of a family of open subsets. Let f be a k-valued function on U. Then f is regular on U iff the restriction $f|_{U_\alpha}$ is regular on each U_α.

(b) Let f be a regular function on an open subset U. Then $D(f) \equiv \{u \in U | f(u) \neq 0\}$ is open and $\frac{1}{f}$ is regular on $D(f)$.

Thus (a) says that a function is regular iff it is locally regular. Also we may add, subtract, multiply and divide regular functions whenever it is reasonable and constant functions are regular.

A first example of a space with functions is:

Example. If $k = \mathbb{R}$ or $\mathbb{C}$ and X is any topological space, then X has a natural structure of a space with functions. Just take $k[U]$ as the set of all continuous functions: $U \to k$.

In algebraic geometry $k[U]$ is commonly denoted by $\mathcal{O}_X(U)$. In general we shall denote a space with functions by the topological space X with the rings of locally regular functions being understood.

A second example of a space with functions is an *open subspace* of a space with functions. Let V be an open subset of a space with functions X. Give V the subspace topology. Then if U is an open subset of V then a regular function on U for the structure of V is simply a regular function on U for the structure on X; i.e., $\mathcal{O}_V(U) = \mathcal{O}_X(U)$.

A *morphism* $f : X \to Y$ between two spaces with functions is a continuous mapping which pulls-back regular functions into regular functions; i.e., if $g(v)$ is a regular function on an open subset V of Y then $f^*(g)(u) \equiv g(f(u))$ is a regular function on the open subset $f^{-1}(V)$ of X . Thus pulling-back by f defines a k-algebra homomorphism $f^* : k[V] \to k[f^{-1}V]$ for each open subset V of Y. An *isomorphism* is a bijective mapping f such that both f and f^{-1} are morphisms.

Exercise 1.1.1. In the first example show that any continuous mapping $X_1 \to X_2$ is a morphism.

Exercise 1.1.2. Prove that the identity of a space with functions is a morphism and the composition of morphisms is a morphism.

Exercise 1.1.3. Let U be an open subspace of a space with functions X. Then show that the inclusion $i : U \to X$ is a morphism and if Y is another space with functions and $g : Y \to U$ is a mapping, then g is a morphism if and only if $i \circ g : Y \to X$ is.

Exercise 1.1.4. Let $f : X \to Y$ be a mapping between spaces with functions. Given open covers $X = \bigcup U_\alpha$ and $Y = \bigcup V_\alpha$ such that $f(U_\alpha) \subseteq V_\alpha$, f is a morphism if and only if each $f_\alpha : U_\alpha \to V_\alpha$ is.

1.2 Varieties

We shall henceforth assume that the field k is algebraically closed.

Let X and Y be spaces with functions. The global effect of pull-back defines a mapping

$$\star : \text{Morphism}(X, Y) \to k\text{-Alg-Hom}(k[Y], k[X])$$

which sends a morphism f to f^*.

An *affine variety* Y is a space with functions such that $\star$ is bijective for every X and $k[Y]$ is a finitely generated k-algebra. An (algebraic) *variety* X is a space with functions X which has a finite open covering $U_1, \ldots, U_n$ where each U_i is affine. A *morphism* of varieties is just a morphism of spaces with functions.

In this section we will give the first examples of varieties: the *affine line* $\mathbb{A}^1$ and the *projective line* $\mathbb{P}^1$.

As a set $\mathbb{A}^1 = \{(x)\}$ is just k. The closed subsets of $\mathbb{A}^1$ are the whole $\mathbb{A}^1$ and the finite subsets. This gives $\mathbb{A}^1$ its topology. Let U be an open subset of $\mathbb{A}^1$. If U is empty, $k[U] = \{0\}$. Otherwise if $U = \mathbb{A}^1 - \{x_1, \ldots, x_n\}$ then $k[U]$ consists of the rational functions $g(x)$ in the coordinate x such that g has no poles anywhere in U; *i.e.* $g(x) = \frac{p(x)}{\prod(x-x_i)^{m_i}}$ where p is a polynomial and the m_i non-negative integers. In particular $k[\mathbb{A}^1] = k[X]$ is a polynomial ring in one variable. We leave the details of checking that $\mathbb{A}^1$ is a space with functions as a very instructive exercise. Here we will check that $\mathbb{A}^1$ is affine.

Let Y be any space with functions. A mapping $f : Y \to \mathbb{A}^1 = k$ is just a k-valued function.

Claim. f is a morphism if and only if f is regular on Y.

Proof. "Only if" is obvious because the function $f = f^\star$(coordinate function x). Conversely, we need to see that if f is regular then f is a morphism; i.e., $f^{-1}(\{x_1, \ldots, x_n\})$ is closed but $f^{-1}(\{x_1, \ldots, x_n\})$ is the complement of the open subset $D(\prod(f - x_i))$ of Y. To show that f pulls-back regular functions, just note $f^\star g = \frac{p(f(y))}{\prod(f(y)-x_i)^{m_i}}$ is regular off $f^{-1}(\{x_1, \ldots, x_n\})$ where g is as before.

To finish we have $\star$: Morphism$(Y, \mathbb{A}^1) \approx k[Y] \approx k$-Alg-Hom $(k[\mathbb{A}^1], k[Y])$. This gives the proof that $\mathbb{A}^1$ is affine. □

As a set $\mathbb{P}^1 = k \amalg \{\infty\}$ where ∞ is a symbol. The non-trivial closed subsets of $\mathbb{P}^1$ are again finite. A regular function on $\mathbb{P}^1 - \{x_1, \ldots, x_n\}$ is a rational function of x which *has no poles* except at $x_1, \ldots, x_n$ where $g(x)$ has no pole at $\infty \equiv g(\frac{1}{y})$ has no pole at $y = 0$. To see that $\mathbb{P}^1$ is a variety it has an open covering $(\mathbb{P}^1 - \{\infty\}) \cup (\mathbb{P}^1 - \{0\})$ by two affine lines. Here $\mathbb{A}^1 \approx \mathbb{P}^1 - \{\infty\}$ sends x to x and $\mathbb{A}^1 \approx \mathbb{P}^1 - \{0\}$ sends x to $1/x$. An interesting feature of this example is:

Exercise 1.2.1. $k[\mathbb{P}^1] = k$.

Exercise 1.2.2. Prove in detail that $\mathbb{A}^1$ is a space with functions.

Exercise 1.2.3. Why is $\mathbb{P}^1$ not affine?

Exercise 1.2.4. Let X and Y be two affine varieties. Show that X is isomorphic to Y iff the k-algebras $k[X]$ and $k[Y]$ are isomorphic.

1.3 The existence of affine varieties

If you did Exercise 1.2.2 you may have already noticed that it is not completely trivial to check that a space with functions is affine. The main work of this chapter is to explicitly construct all affine varieties. The main result is

Theorem 1.3.1. *If A is a finitely generated k-algebra with no nilpotents, then there is a canonically constructed affine variety* Spec A *with a natural isomorphism*

$$A \approx k[\text{Spec } A].$$

Recall that A has no nilpotents if $a^n = 0$ for some $n > 0$ implies $a = 0$ in A. Clearly a ring of k-valued functions has no nilpotents. Therefore the hypothesis of the theorem is necessary for A to be k[affine variety]. So the theorem constructs all affine varieties up to isomorphism (see Exercise 1.2.4).

In this section we will define Spec A as a space with functions together with the homomorphism $\phi : A \to k[\text{Spec } A]$. The rest of this chapter is devoted to the proof that

($\star$) Spec A is affine and
($\star\star$) ϕ is an isomorphism.

As a set Spec A = k-Alg-Hom (A, k). We have a natural k-algebra homomorphism $\phi : A \to \{k\text{-valued functions on Spec } A\}$. Just let $\phi(a)(x) \equiv x(a)$ where x is a point of Spec A.

Let I be a subset of A. Define $\text{zeroes}(I) = \{x \in \text{Spec } A | i(x) = 0$ for all $i \in I\}$.

Claim. The subsets $\{\text{zeroes}(I)\}_{I \subset A}$ are the closed subsets of a topology of Spec A.

Proof.

(*a*) Spec A = zeroes($\{0\}$) and $\emptyset$ = zeroes($\{1\}$).
(*b*) $\text{zeroes}(I_1 \cdot I_2) = \text{zeroes}(I_1) \bigcup \text{zeroes}(I_2)$.
(*c*) $\text{zeroes}(\bigcup I_i) = \bigcap \text{zeroes}(I_i)$. Why?

□

If U is an open subset of Spec$A$, a regular function on U is a k-valued function f on U such that there is an open covering $U = \bigcup U_i$ such that each $f|_{U_i}$ has the form $\frac{\phi(a_i)(x)}{\phi(b_i)(x)}$ where the denominator $\phi(b_i)(x)$ never vanishes on U_i. Clearly $k[U]$ is a k-algebra and condition (a) is easily verified. For condition (b) note that $D(f) = \bigcup(U_i - \text{zeroes}(a_i) \bigcap U_i)$ is open and $1/f \left(= \frac{\phi(b_i)x}{\phi(a_i)x}\right)$ is regular on $D(f)$. As $\phi(a)(x) = \frac{\phi(a)(x)}{\phi(1)(x)}$ the image of ϕ is contained in $k[\text{Spec}A]$, that is $\phi : A \to k[\text{Spec } A]$.

Exercise 1.3.2. Check the above details.

Exercise 1.3.3. Show that, for any ring A, the set of nilpotent elements $\sqrt{0} = \{a \in A | a^n = 0 \text{ for some } n > 0\}$ is an ideal. Also check that $A = \sqrt{0} \iff A = \{0\}$.

Exercise 1.3.4. Set-theoretically, what is Spec A when A is the polynomial ring $k[X_1, \ldots, X_n]$? Same if $A = k[X_1, \ldots, X_n]/(f_1, \ldots, f_n, \ldots,)$ for some polynomials f_i.

Exercise 1.3.5. If A is a finitely generated ring with no nilpotents, let a be an element of A. Consider the homomorphism $\phi : A \to A_{(a)}$, where $A_{(a)}$ is a finitely generated k-algebra with no nilpotents (Why?). Show that ϕ induces an isomorphism $\text{Spec}(A_{(a)}) \overset{\approx}{\to} D(a) \subset \text{Spec } A$ where $D(a)$ is given the open subspace structure as a space with functions.

Exercise 1.3.6. Show that the collection $\{D(a)\}_{a \in A}$ are a basis for the topology of Spec A and $D(a_1) \cap D(a_2) = D(a_1 \cdot a_2)$.

1.4 The nullstellensatz

The objective of this section is to prove Hilbert's nullstellensatz which says that Spec A has enough points so that

$$(**)_1 \qquad \phi : A \to k[\text{Spec } A]$$

is injective.

We will begin with a lemma of E. Noether whose proof will be presented in the second chapter.

Lemma 1.4.1. *Let A be a non-zero finitely generated k-algebra. Then we have an injection $B \subset A$ where B is a polynomial ring $k[X_1, \ldots, X_d]$ such that A is a B-module of finite type.*

Next we have

Lemma 1.4.2. *Let $k \subset B \subset A$ be k-algebras such that A is a B-module of finite type. Then composition gives a surjection*

$$k\text{-Alg-Hom}(A, k) \to k\text{-Alg-Hom}(B, k).$$

Proof. Let $\psi : B \to k$ be a k-algebra homomorphism. Let m be the kernel of ψ. Consider the ideal mA in A. If $mA \neq A$, we can find a maximal ideal n of A such that $n \supset mA$. Then A/n is a finite field extension of $B/m = k$. Thus $A/n = k$ as k is algebraically closed and the quotient mapping $A \to k$ is an extension of ψ.

It remains to show that $mA = A$ is impossible. Let $A = B^r/R$ as a B-module where R is a B-submodule of B^r where r is finite. Let $e_1, \ldots, e_r$ be the unit vectors in B^r. Then for $1 \leq i \leq r$, $e_i \in mB^r + R$, i.e., $e_j = \sum_k m_j^k e_k + r_j$ where the r_j are in R and the m_j^k are in m. So if $(b_j^k) \equiv 1_r - (m_j^k)$ then $r_j = \sum b_j^k e_k$ is in R. By Cramer's rule $\det(b_j^k)e_i$ is a linear combination of the r_j with coefficients in B. Thus $\det(b_j^k)B^r \subseteq R$ and hence $\det(b_j^k) \cdot A = 0$, or, what is the same, $\det(b_j^k) = 0$. On the other hand $\det(b_j^k) = 1((m))$. As $1 \neq 0((m))$ we have a contradiction. □

With these lemmas the proof of the nullstellensatz is easy. Let a be a non-zero element of A. Consider the localization $A_{(a)}$. This is non-zero as $1/1 = 0/1 \Leftrightarrow a^n = a^n \cdot 1 = a^n \cdot 0 = 0$ for $n > 0$ and A has no nilpotents. Furthermore $A_{(a)}$ is generated by A and $1/a$ hence is finitely generated. Thus by Lemma 1.4.1 we have an inclusion $k[X_1, \ldots, X_r] \subset A_{(a)}$.

We need to find a point x of Spec A such that $\phi(a)(x) \equiv x(a)$ is non-zero where $x : A \to k$ is a k-algebra homorphism. Let $z : k[X_1, \ldots, X_r] \to k$ be any k-algebra homomorphism. By Lemma 1.4.2 we may lift z to a k-algebra homomorphism $y : A_{(a)} \to k$. Let x be the composition $A \to A_{(a)} \xrightarrow{y} k$. Then $1 = x(1) = y(a/a) = y(a/1)y(1/a) = x(a)y(1/a)$. Therefore $x(a) \neq 0$. This proves the nullstellensatz.

Remark. The argument with Cramer's rule in the last part of Lemma 1.4.2 can be generalized to prove

Lemma 1.4.3. (Nakayama.) *Let M be a finitely generated module over a ring A. Assume that there is an ideal I of A such that $M = I \cdot M$. Then there is an element a of $1 + I$ such that $aM = \{0\}$.*

Exercise 1.4.4. Prove Nakayama's lemma.

Henceforth we shall identify A with a ring of functions of Spec A. Next we will give a reformulation of the nullstellensatz which resembles Hilbert's original statement.

Theorem 1.4.5. (Nullstellensatz.) *If I is an ideal of A, $\{a \in A | a(x) =$ 0 for all x in zeroes$(I)\} = \sqrt{I}$ where $\sqrt{I} = \{a \in A | a^n \in I$ for some $n > 0\}$.*

Proof. One checks that $\sqrt{I}$ is an ideal. Then $A' = A/\sqrt{I}$ is a finitely generated ring which has no nilpotents. Let b be an element of $A - \sqrt{I}$. Then $b' = b + \sqrt{I}$ is a non-zero element of A'. By $(**)_1$, for A' we have a homomorphism $x' : A' \to k$ such that $x'(b') \neq 0$. Let $x : A \to A' \xrightarrow{x'} k$ be the composition. Then by construction x is a point of zeroes(I) and $b(x) \neq 0$. This proves the desired inclusion "$\subseteq$". The reverse inclusion is trivial because $b^n(x) = 0 \Leftrightarrow b(x) = 0$ if $n > 0$. □

Some special cases will be useful.

Corollary 1.4.6. *Given a subset J of A then* zeroes $(J) = \emptyset \Leftrightarrow 1 = \sum_{\text{finite}} a_k j_k$ *where $a_k \in A$ and $j_k \in J$.*

Proof. Apply Theorem 1.4.5 to the ideal $I = JA$. Hence if zeroes$(J) = \emptyset$, then zeroes$(I) = \emptyset$ and we have $A = \sqrt{I}$; i.e., $1 = 1^n \in JA$ which is the second statement. The converse is obvious. □

Corollary 1.4.7. *Let f_i be elements of A and n_i be positive integers.*

$$\text{Spec}(A) = \bigcup D(f_i) \Leftrightarrow 1 = \sum_{\text{finite}} a_i f_i^{n_i}.$$

Proof. The complement of the open subset $\bigcup D(f_i)$ is zeroes$\{(f_i^{n_i})\}$. Thus this corollary follows from the last. □

This result has an interesting topological consequence.

Corollary 1.4.8. *A variety is quasi-compact.*

Proof. As a variety is a finite union of open affines, it suffices to prove this for the affine variety Spec A. Now the $D(f_i)$ are a basis for the topology of Spec A. If Spec $A = \bigcup D(f_i)$ is an open cover by this then

$1 = \sum a_i f_i$ for f_i in a finite set I of indexes. Then Spec $A = \bigcup_{i \in I} D(f_i)$ is a finite subcover. □

Exercise 1.4.9. Prove that a point of a variety is a closed subset. (Hint: reduce to the affine case.)

1.5. The rest of the proof of existence of affine varieties / subvarieties

We will first show

$(**)_2$ The subring $A \subset k[\text{Spec } A]$ is all of $k[\text{Spec } A]$.

Let $f(x)$ be a function in $k[\text{Spec } A]$. We need to see that f is in A. By definition we have an open cover Spec $A = \bigcup U_\alpha$ by open subsets such that $f(u) = \frac{g_\alpha(u)}{h_\alpha(u)}$ when u in U_α where g_α and h_α are in A and $h_\alpha(u)$ is never zero on U_α. We may assume that $U_\alpha = D(k_\alpha)$ for some k_α in A. Doing the replacement $f(u) = \frac{(k_\alpha g_\alpha)(u)}{(k_\alpha h_\alpha)(u)}$ on $D(k_\alpha \cdot h_\alpha) = D(k_\alpha)$, we may assume that $h_\alpha = k_\alpha$.

Next consider the function $h_\alpha^2 f$. This equals $h_\alpha g_\alpha$ on $D(h_\alpha)$ and both functions are zero on the complement. Therefore $h_\alpha^2 f = h_\alpha g_\alpha$ is in A. By Corollary 1.4.7, $1 = \sum a_\alpha h_\alpha^2$ for some a_α in A. Thus $f = f \cdot 1 = \sum a_\alpha (f h_\alpha^2)$ is in A, which is what we wanted.

It remains to prove $(*)$.

Let X be a space with functions. Then $\star$ defines a bijection Morphism(X,Spec A) $\to$ k-Alg-Hom$(A, k[X])$. Let δ_x be evaluation of a function at a point x. Let $f : X \to \text{Spec} A$ be a morphism. Let a be an element of A and x be a point of X. Then $a(f(x)) = (f^\star a)(x)$, or, rather, $\delta_{f(x)}$ is the composition $A \xrightarrow{f^\bullet} k[X] \xrightarrow{\delta_x} k$. As the point $f(x)$ is the same as the homomorphism $\delta_{f(x)}$, this shows that f is determined by $f^\star$. Thus $\star$ is injective. Conversely let $\phi : A \to k[X]$ be a k-algebra homomorphism. Define $f \equiv r(\phi)$ by the formula $\delta_{f(x)} = \delta_x \circ \phi$. Using properties (a) and (b) of a space with functions, you do

Exercise 1.5.1. f is a morphism.

Clearly r is an inverse to $\star$. Thus $(*)$ is true and we have constructed all affine varieties up to isomorphism.

Exercise 1.5.2. Show that the category of affine varieties with morphisms is contravariantly equivalent to the category of finitely generated k-algebras with no nilpotents with k-algebra homomorphisms.

In general let X be a subset of a space with functions Y. Then X has an *induced structure of a space with functions.* Explicitly give X the subspace topology. A regular function f on an open set U of X is a function of the following form; there is an open cover $U = \bigcup(X \cap V_\alpha)$ where the V_α are open subsets of Y such that $f(y) = g_\alpha(y)$ on $X \cap V_\alpha$ where g_α is regular on V_α.

Exercise 1.5.3. Check that X is a space with functions and the inclusion $X \hookrightarrow Y$ is a morphism.

With this notation we have

Theorem 1.5.4. *A locally closed subspace of a variety is a variety called a* subvariety*. A closed subspace of an affine variety is affine and in this case a regular function on the subspace lifts to a regular function on the ambient variety.*

Proof. Let X be a locally closed subset of a variety Y. Then X is a closed subset of an open subset Z of Y. Clearly the space with function structures on X induced by Z and Y are the same. Thus we need to know the two cases X open in Y and X closed in Y.

Let $Y = \bigcup Y_i$ where the Y_i are a finite number of open affines. Then $X = \bigcup(X \cap Y_i)$ is a finite open cover of X. As the statements are local on Y we may assume that Y is affine.

Then we want to prove

(a) if X is closed then X is affine,
(b) if X is open then X has a finite covering by open affines.

Now let $Y = \text{Spec } A$ where A is a finitely generated k-algebra with no nilpotents. Assume that X is closed. Let I be the ideal of functions in A vanishing identically on X. Clearly set-theoretically $X = \text{Spec}(A/I)$. One simply checks directly from the definitions that

Exercise 1.5.5. $\text{Spec}(A/I)$ has the induced structure as a space with functions. Therefore X is affine if it is closed in an affine.

Assume that X is open. Then $X = \bigcup D(a_i)$ where the a_i are in A. Then the $D(a_i)$ are the affines $\text{Spec}(A_{(a_i)})$ by Exercise 1.3.5. To see that there are only finitely many necessary $D(a_i)$ we use

Lemma 1.5.6. *Any open subset of a variety is quasi-compact.*

This will be proved in the second chapter.

Exercise 1.5.7. Let X be a subspace of a space with functions Y. For any space with functions Z a mapping $Z \to X$ is a morphism if the composition $Z \to X \to Y$ is a morphism.

1.6 $\mathbb{A}^n$ and $\mathbb{P}^n$

By definition $\mathbb{A}^n = \mathrm{Spec}(k[X_1, \ldots, X_n])$. As a k-algebra homomorphism $k[X_1, \ldots, X_n] \to k$ is determined by the images $x_1, \ldots, x_n$ of $X_1, \ldots, X_n$ in k which can be arbitrary, $\mathbb{A}^n = \{(x_1, \ldots, x_n) \in k^n\}$ set-theoretically. As a variety $\mathbb{A}^n$ is called an *affine n-space*.

Furthermore if X is a space with functions and $f : X \to \mathbb{A}^n$ is a mapping given by $f(x) = (f_1(x), \ldots, f_n(x))$ then f is a morphism if and only if each $f_i(x)$ is a regular function on X.

Another fact is that any affine variety X is isomorphic (non-canonically) to a closed subvariety of $\mathbb{A}^n$ corresponding to a surjection $k[X_1, \ldots, X_n] \to A$ where $A = k[X]$. A subvariety of $\mathbb{A}^n$ (or any other affine variety) is called *quasi-affine*.

For instance the punctured affine space $\mathbb{A}^n - \{0\}$.

Lemma 1.6.1.

(*a*) $k[\mathbb{A}^1 - \{0\}] = k[X_1, X_1^{-1}]$.
(*b*) *If* $n > 1$, $k[\mathbb{A}^n - \{0\}] = k[X_1, \ldots, X_n]$.

Proof. If $n = 1$, $\mathbb{A}^1 - \{0\}$ is the open subvariety $D(X_1)$ of $\mathbb{A}^1$. So $\mathbb{A}^1 - \{0\}$ is the affine $\mathrm{Spec}(k[X]_{(X)} = k[X, X^{-1}])$. Thus $k[X, X^{-1}] = k[\mathbb{A}^1 - \{0\}]$.

If $n \geq 2$, $A^n - \{0\} = D(X_1) \cup \ldots \cup D(X_n)$ and

$$k[D(X_i)] = k[X_1, \ldots, X_n, X_i^{-1}].$$

Thus $k[\mathbb{A}^n - \{0\}] = \bigcap_i k[X_1, \ldots, X_n, X_i^{-1}] = k[X_1, \ldots, X_n]$. □

Exercise 1.6.2. Prove that the quasi-affine variety $\mathbb{A}^2 - \{0\}$ is not affine.

Set-theoretically $\mathbb{P}^n$ is the quotient set

$$(\mathbb{A}^{n+1} - \{0\})/k^* = \{(x_0, \ldots, x_n) \neq 0$$

modulo $(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n)$ for any λ in $k - \{0\}\}$. We want to define $\mathbb{P}^n$ as a quotient space with functions. Let $\pi : \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$ send a vector to all vectors with the same direction. A subset U of $\mathbb{P}^n$ is open iff $\pi^{-1}U$ is open. A function f on U is regular iff $\pi^* f$ is regular on $\pi^{-1}U$. We claim that with this definition of a space with functions $\mathbb{P}^n$ is a variety. Let $0 \leq i \leq n$. Let $E_i = \{(x_0, \ldots, x_n)$ in $\mathbb{P}^n \mid x_i \neq 0\}$.