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Manifolds with Singularities and the Adams-Novikov Spectral Sequence

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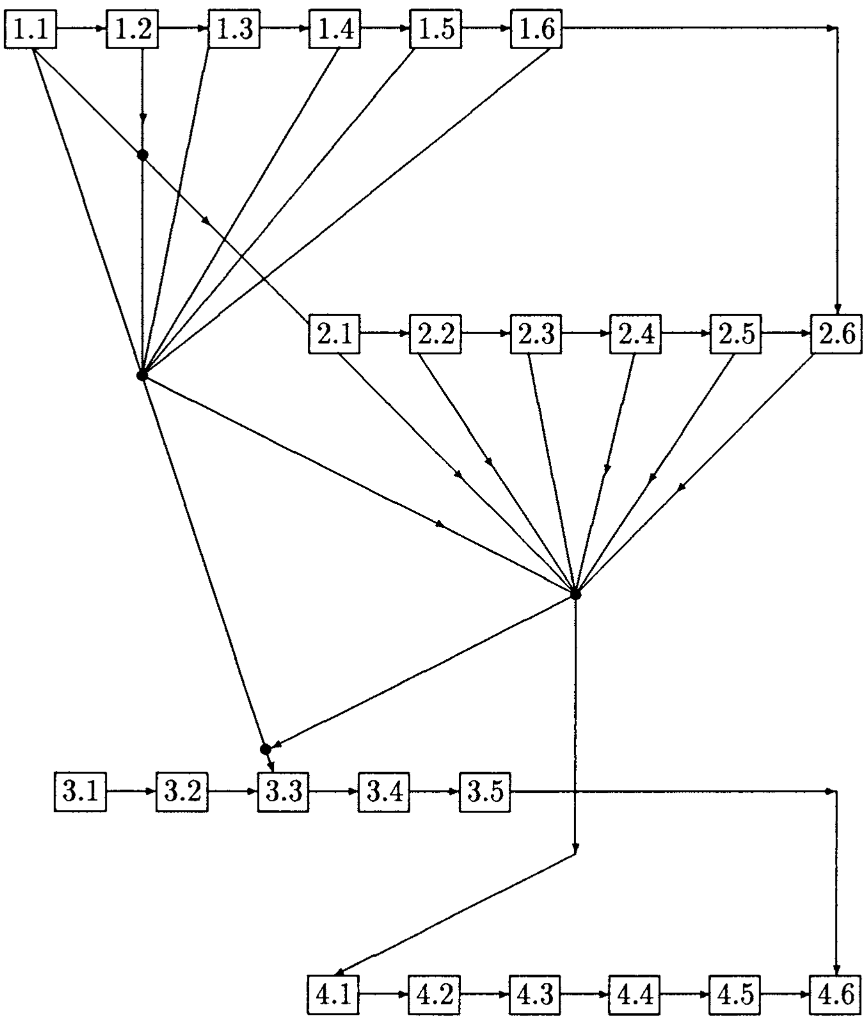

Preface

The purpose of this book is to discuss some natural relations between geometric concepts of Cobordism Theory of manifolds with singularities and the Adams-Novikov spectral sequence.

We begin by motivating this discussion. The central problem of Algebraic Topology has been and continues to be that of obtaining geometrically manageable descriptions of the main algebraic invariants and constructions. For example, let us note the Steenrod problem on realization of integer homology cycles by manifolds. When cycles are presented in terms of manifolds we are able to deal with the cycles by means of Smooth or Piecewise Linear Topology, to apply surgery, to paste and cut and so on. A nice example of successful combining of algebraic and geometric methods may be found in Sullivan's approach to the Hauptvermutung [105]. In particular, Sullivan discovered new geometric objects of Algebraic Topology, namely *manifolds with singularities*. The simplest example is a $(\mathbf{Z}/n)$-manifold (see Figure 0.1, where $n = 4$).

$(\mathbf{Z}/n)$-manifolds present homology cycles with $\mathbf{Z}/n$ coefficients allowing us to apply the machinery of smooth topology. The corresponding (co)-bordism theory naturally represents ordinary (co)-homology theory with coefficients in $\mathbf{Z}/n$.

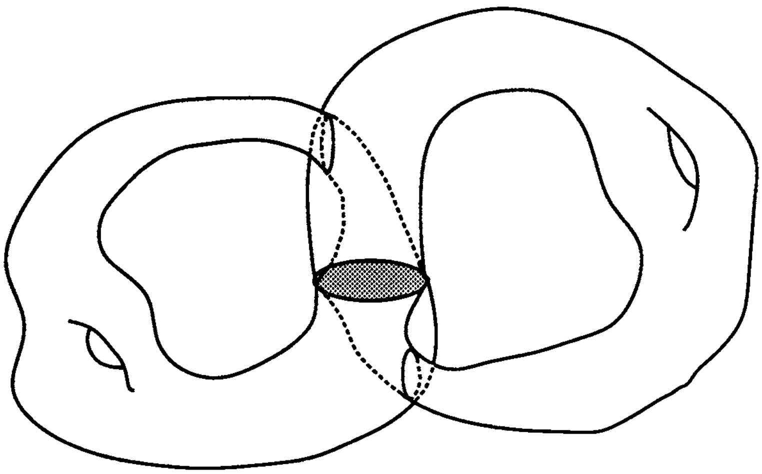

Figure 0.1: $\mathbf{Z}/4$-manifold.

In the early seventies D.Sullivan [106] and N.Baas [11] defined the bordism and cobordism theories of manifolds with singularities in the general case. Then it was made evident that many known homology theories may be realized as bordism theories with singularities. The notion of manifolds with singularities allows us to geometrize the complex and real K-theories, Morava K-theories [93], [115], [116], the Conner-Floyd theory $W(\mathbf{C}, 2)_*(\cdot)$ [67] and some others whose cycles were not provided with some geometric structure.

Although the notion of *manifold with singularities* is very close to the notion of *ordinary manifold* we have to be much more careful when dealing with the former. For example, the direct product of two manifolds with singularities doesn't possess the structure of such a manifold. Actually there exist some manifolds with singularities which are obstructions to the existence of an *admissible product*. In particular the problem concerning multiplicativity of the ordinary homology theory with $\mathbf{Z}/n$ coefficients may be solved in the same terms. O.K.Mironov [67], [68] was the first to give the general geometric approach to multiplicativity of bordism theories with singularities; see also [9], [16], [70], [116]–[119]. The main constructions pertaining to multiplicativity are described in Chapter 2 of this book.

Now *manifold with singularities* is a common and convenient notion as well as *ordinary manifold.* The corresponding cobordism and bordism theories have been considered from several viewpoints. The connection between the complex cobordism theory with singularities and Formal Group Theory has been found (see [48], [51], [93], [94], [115]–[121]); now a new interesting application has appeared, namely the Theory of Elliptic Genera.

It isn't our purpose to describe all the rich applications of bordism theories with singularities. We would rather concentrate our attention on the connection between geometry of manifolds with singularities and the Adams-Novikov spectral sequence.

Traditionally the Adams-Novikov spectral sequence (ANSS) has been considered as a computational machine allowing us to describe the stable homotopy groups $\pi_* X$ of the given spectrum X in algebraic terms of generators and relations. Actually it can be seen that geometric methods are widely used for description and computation of the Adams-Novikov spectral sequences. For example, the concepts of J-homomorphism and Hopf invariant which originally were geometric tools have been transformed to the powerful chromatic machinery; see [35], [48], [61], [82]–[85]. The chromatic technique allows us to subdivide the Adams-Novikov spectral sequence for a sphere into parts, each of which is determined by the corresponding Morava K-theory $k\langle n\rangle^*(\cdot)$. So the computational problem is reduced to some particular algebraic and homotopy problems.

We note that consideration of the Adams-Novikov spectral sequence for spheres is not the subject of this book. For detailed information refer to D.Ravenel's book [84].

The main topological object which is going to be considered here is the symplectic cobordism ring MSp_*. Actually we are going to examine the Adams-Novikov spectral sequence for this ring. We shall not regard *ANSS* as a computational tool only, but as a mathematical object provided with rich algebraic and geometric structures. Particular attention will be paid to finding and describing the above geometric

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structure.

V.Vershinin has gone rather far in the computation of this Adams-Novikov spectral sequence [109]–[113]. Our intention is to use his results and constructions widely.

Geometric methods are also very useful to deal with the Adams-Novikov spectral sequence for the ring MSp_*. For example, Two-valued Formal Group Theory (see V.Buchstaber [26], [27]) describes the ring

$$Л_* = Hom^*_{\mathbf{A}^{MU}}\left(MU^*(MSp), MU^*\right),$$

which is the zero line of $E_2^{*,*}$, in various terms. We emphasize that application of Two-valued Formal Group Theory is of substantial help to compute this spectral sequence; see [109]–[113].

The first necessary step to start dealing with the Adams-Novikov spectral sequence for a spectrum X is to present a particular *Adams resolution* for this spectrum. Conventionally this is constructed in a standard category of spectra. Sometimes the Adams resolution as well as the Adams-Novikov spectral sequence can be constructed directly from geometric consideration of some notions of cobordism with singularities. To clarify the above statement we consider the following.

Example. The *Adams resolution for the spectrum MSU* can be constructed as follows. Suppose θ_1 is the generator of the group $MSU_1 = \mathbf{Z}/2$, and P is the framed circle presenting the element θ_1 . According to Mironov [67], the bordism theory with θ_1-singularity $MSU^*_{\theta_1}(\cdot)$ is isomorphic to the *Conner-Floyd cohomology theory* $W(\mathbf{C},2)^*(\cdot)$ (see Stong [103]). We note that the Bockstein-Sullivan exact sequence

$$\cdots \longrightarrow MSU^*(\cdot) \xrightarrow{\cdot\theta_1} MSU^*(\cdot) \xrightarrow{\pi} MSU^*_{\theta_1}(\cdot) \xrightarrow{\partial} MSU^*(\cdot) \to \cdots$$

induces the diagram of the classifying spectra:

$$\begin{array}{ccccccc} MSU & \xleftarrow{\cdot\theta_1} & MSU & \xleftarrow{\cdot\theta_1} & MSU & \xleftarrow{\cdot\theta_1} & \cdots \\ & {\searrow^{\pi}} \quad {\nwarrow^{\partial}} & & {\searrow^{\pi}} \quad {\nwarrow^{\partial}} & & {\searrow^{\pi}} & \\ & MSU^{\theta_1} & \xrightarrow{\mathfrak{b}} & MSU^{\theta_1} & \xrightarrow{\mathfrak{b}} & \cdots & \end{array} \qquad (0.1)$$

It is obvious that the diagram (0.1) is an Adams resolution of the spectrum MSU in the cohomology theory $MSU^*_{\theta_1}(\cdot)$. In the 2-local

local category this diagram also presents a particular Adams resolution in the Brown-Peterson theory $BP^*(\cdot)$ since the spectrum $MSU_{(2)}^{\theta_1}$ splits into a wedge of the spectra $\Sigma^n BP$.

A complete description of the bordism ring MSU_* in terms of generators and relations was obtained as a result of considering the above Adams-Novikov spectral sequence from geometric and algebraic viewpoints; see [14].

It can be seen now that the Adams-Novikov spectral sequence may be determined by a procedure of resolving singularities. □

It is natural to suppose that such a procedure does exist in the case of several singularities as well. Indeed every given bordism theory $MG_*(\cdot)$ and sequence $\Sigma = (P_1, \cdots, P_k, \cdots)$ of closed manifolds naturally determine the theories $MG_*^{\Sigma\Gamma(k)}(\cdot)$ which are interconnected and are related to the theories $MG_*(\cdot)$, $MG_*^{\Sigma}(\cdot)$ as can be seen in the following diagram:

$$\begin{array}{ccccccccc} MG_* & & \xleftarrow{\ \gamma(1)\ } & & MG_*^{\Sigma\Gamma(1)} & & \xleftarrow{\ \gamma(2)\ } & & MG_*^{\Sigma\Gamma(2)} \xleftarrow{\gamma(3)} \cdots \\ & \searrow^{\pi(0)} & & \nearrow^{\partial(1)} & & \searrow^{\pi(1)} & & \nearrow^{\partial(2)} & \quad \searrow^{\pi(2)} \\ & & MG_*^{\Sigma(1)} & & \xrightarrow{\ \beta(1)\ } & & MG_*^{\Sigma(2)} & & \xrightarrow{\ \beta(2)\ } \cdots \end{array} \tag{0.2}$$

Here the theories $MG_*^{\Sigma(k)}(\cdot)$ split into the sums of the theories $MG_*^{\Sigma}(\cdot)$ and the transformations $\gamma(k)$, $\pi(k)$, $\partial(k)$ will be defined in geometric terms of cutting and gluing the manifolds. For example, the *manifold M in the bordism theory* $MG_*^{\Sigma\Gamma(1)}(\cdot)$ is glued out of the blocks $\gamma_i(M) \times P_i$ in the same way as a *boundary* of a closed manifold with singularities is glued after removing the cones over singularities.

The diagram (0.2), being an exact couple, induces the spectral sequence (it will be called the Σ-*singularities spectral sequence*). This spectral sequence restores the bordism theory $MG_*(\cdot)$ out of the bordism theory with singularities $MG_*^{\Sigma}(\cdot)$.

The top line of the diagram (0.2) represents the filtration of the bordism theory $MG_*(\cdot)$, which may be considered as a geometric analogy of the algebraic filtration generated by powers of the ideal

$$\mathfrak{m} = ([P_1], \cdots, [P_k] \cdots) \subset MG_*.$$

The differentials in the Σ-singularities spectral sequence also have a simple geometric description. The first differential is a direct sum of the Bockstein operators β_k (which are similar to boundary operators on ordinary manifolds). It is very important for our purposes that in several cases the Σ-singularities spectral sequence may be naturally identified with the corresponding Adams-Novikov spectral sequence.

Now let us briefly describe the subject of every chapter. Chapter 1 contains basic geometric constructions of manifolds with singularities. The definition and properties of the Σ-singularities spectral sequence are given at the end of the chapter.

In Chapter 2 we give a geometric construction of a multiplication on bordism theories with singularities. The chapter provides a necessary geometric tool for algebraic considerations.

Chapter 3 is concerned with algebra. First of all we deal with definitions of the Adams-Novikov spectral sequence *(ANSS)* and the Novikov algebraic spectral sequence. Next the symplectic bordism theory with singularities $MSp^{\Sigma}_*(\cdot)$, which has been discovered by V.Vershinin [111], is considered. The coefficient ring MSp^{Σ}_* of this theory is a polynomial ring and the theory $MSp^{\Sigma}_*(\cdot)_{(2)}$ splits into a direct sum of Brown-Peterson theories $BP^*(\cdot)$. So we can identify the corresponding Adams-Novikov spectral sequence with the Σ-singularities spectral sequence. The proof of Vershinin's theorem concerning the theory $MSp^{\Sigma}_*(\cdot)$ is given, since its details will be applied later.

It can be seen that the first differential in the Adams-Novikov spectral sequence splits into a sum of Bockstein operators. So we have a new way to compute the algebra

$$Ext^{*,*}_{\mathbf{A}^{BP}}(BP^*(MSp), BP^*).$$

Some computation is given in Chapter 4. The product structure in the theory $MSp^{\Sigma}_*(\cdot)$ is carefully chosen and the action of Bockstein operators on the generators of the coefficient ring MSp^{Σ}_* is computed.

The result is rather surprising. Indeed, the algebra

$$Ext^{*,*}_{\mathbf{A}^{BP}}(BP^*(MSp^{\theta_1}), BP^*)$$

has a module structure over the symmetric group, so it may be described in terms of representation theory.

So we have tried to come along the way from simple geometric considerations concerning manifolds with singularities up to some computational results describing the structure of the Adams-Novikov spectral sequence.

We use many figures hoping they may be helpful to understand the discussions and arguments.

The list of references doesn't claim completeness.

It is a pleasure to acknowledge Victor M. Buchstaber for helpful comments and valuable critical notes and Vladimir V. Vershinin, Vassily G. Gorbunov, Roin G. Nadiradze for fruitful collaboration and discussions on the subject of this book.

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Boris Botvinnik