Chapter 1

Manifolds with singularities

The purpose of this chapter is to describe a procedure restoring an ordinary bordism theory $MG_*(\cdot)$ out of the bordism theory with singularities $MG_*^{\Sigma}(\cdot)$. This geometric procedure looks like resolving the singularities in the manner of Cusp Theory.

It is our hope that the constructing of the Σ-singularities spectral sequence (Σ-*SSS*) will not get us bogged down in modern Homological Algebra. The main objects for consideration will be manifolds with various geometric structures. The initial point here is the following simple observation. The given bordism theory $MG_*(\cdot)$ and the sequence $\Sigma = (P_1, \cdots, P_n, \cdots)$ of closed manifolds induce not only the bordism theory with singularities $MG_*^{\Sigma}(\cdot)$, but also the family of intermediate bordism theories $MG_*^{\Sigma\Gamma(k)}(\cdot)$ for $k = 1, 2, \ldots$. A *manifold M in the theory $MG_*^{\Sigma\Gamma(k)}(\cdot)$* (it will be called a $\Sigma\Gamma(k)$-*manifold*) is consistently glued out of the blocks

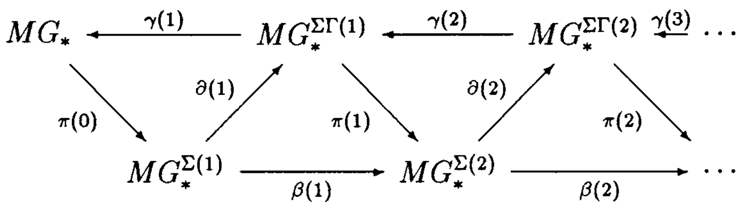
$$\gamma_\alpha M \times (P_1^{a_1} \times \ldots \times P_n^{a_n} \times \cdots),$$

where $\alpha = (a_1, \ldots, a_n, \ldots)$ is a sequence of nonnegative integers such that $a_1 + \ldots + a_n + \ldots = k$, and $\gamma_\alpha M$ are ordinary manifolds. It is evident that the family of theories $MG_*^{\Sigma\Gamma(k)}(\cdot)$ gives us the filtration of the theory $MG_*(\cdot)$:

$$MG_*(\cdot) \xleftarrow{\gamma(1)} MG_*^{\Sigma\Gamma(1)}(\cdot) \longleftarrow \cdots \longleftarrow MG_*^{\Sigma\Gamma(k)}(\cdot) \xleftarrow{\gamma(k+1)} \cdots$$

This filtration generates the Σ-SSS.

Following O.K.Mironov [67], [68] we begin with the definitions of the bordism theories with singularities $MG_*^{\Sigma}(\cdot)$. His constructions seem to have a most clear and obvious form for our purposes. Then we'll define the bordism theories $MG_*^{\Sigma\Gamma(k)}(\cdot)$, $MG_*^{\Sigma(k)}(\cdot)$ from the following diagram in the same manner:



The transformations $\gamma(k)$, $\partial(k)$, $\pi(k)$ will also be defined geometrically, by gluing and cutting the manifolds.

Actually the main part of the chapter (sections 1.2–1.5) contains only some geometric constructions from elementary Cobordism Theory. Almost all the proofs will be given by means of constructing bordisms joining some manifolds. We'll deal with some elementary homological algebra only in section 1.3. We'll define the Σ-SSS and prove its simplest properties.

We note also that the Σ-SSS was defined by V.V.Vershinin [111] in the one-singularity case, and the general case was described by the present author [15], [16].

1.1 Bordism theories with singularities

Here we define the bordism and cobordism theories with singularities. We wouldn't like to consider an absolutely general case; here we shall be restricted to the following situation.

The starting point is a category of smooth manifolds with a stable G-structure in the stable normal bundle, where G is one of the classic Lie groups. The corresponding bordism and cobordism theories will be denoted by $MG_*(\cdot)$ and $MG^*(\cdot)$. Our main examples will be connected with the cobordism theories $MO^*(\cdot)$, $MSO^*(\cdot)$, $MU^*(\cdot)$, $MSU^*(\cdot)$, $MSp^*(\cdot)$.

Also we suppose that a direct product of the manifolds generates some external product structure in the theories $MG^*(\cdot)$ and $MG_*(\cdot)$ with the usual properties (see [67], and section 2.1). The classifying Thom spectrum for these theories will denoted by MG.

Let us take a sequence $\Sigma = (P_1, \ldots, P_n, \ldots)$ of closed manifolds. It is supposed below that the sequence Σ is *locally-finite*, i.e. the number sequence $\{\dim P_n\}$ has only infinity as a point of condensation. We denote $\Sigma_k = (P_1, \ldots, P_k)$ for every $k = 1, 2, \ldots$. It is convenient to denote $P_0 = pt$.

Definition 1.1.1 *We call a manifold M a Σ_k-manifold if there are given the following:*

(i) the partition

$$\partial M = \partial_0 M \cup \partial_1 M \cup \ldots \cup \partial_k M$$

of its boundary ∂M into such a union of manifolds that the intersection

$$\partial_I M = \partial_{i_1} M \cap \ldots \cap \partial_{i_q} M$$

is a manifold for every collection $I = \{i_1, \ldots, i_q\} \subset \{0, 1, \ldots, k\}$ and its boundary is equal to

$$\partial\,(\partial_I M) = \bigcup_j (\partial_I M \cap \partial_j M)\,;$$

(ii) the compatible product structures (i.e. diffeomorphisms preserving the stable G-structure)

$$\phi_I : \partial_I M \longrightarrow \beta_I M \times P^I,$$

where $I = \{i_1, \ldots, i_q\} \subset \{0, 1, \ldots, k\}$,

$$P^I = P_{i_1} \times \ldots \times P_{i_q}.$$

Compatibility here means that if $I \subset J$ *and*

$$\iota : \beta_I M \longrightarrow \beta_J M$$

is the inclusion, then the map

$$\phi_I \circ \iota \circ \phi_J^{-1} : \beta_I M \times P^J \longrightarrow \beta_I M \times P^I$$

is identical on the direct factor P^I. □

Note 1.1.1 *Now we have defined half-finished manifolds with singularities, to obtain real manifolds with singularities we have to do some identification.* □

Two points x, y of the Σ_k-manifold M are *equivalent* if they belong to the same manifold $\partial_I M$ for some $I \subset \{0, 1, \ldots, k\}$ and

$$pr \circ \phi_I(x) = pr \circ \phi_J(y),$$

where

$$pr : \beta_I M \times P^I \longrightarrow \beta_I M$$

is the projection on the direct factor. The factor-space of the topological space M under this equivalence relation is called *the model of the* Σ_k-*manifold* M and is denoted by M_Σ.

Indeed it is convenient to deal with Σ_k-manifolds without considering their models. For this we only have to remember consistency of the constructions with the projection

$$\pi : M \longrightarrow M_\Sigma.$$

The *boundary* δM *of a* Σ_k-*manifold* M is the manifold $\partial_0 M$. It is also a Σ_k-manifold:

$$\partial_I(\delta M) = \partial_I M \cap \delta M.$$

Manifolds $\beta_I M$ are also Σ_k-manifolds:

$$\partial_j(\beta_I M) = \left\{ \begin{array}{ll} \emptyset & \text{if } j \in I, \\ \beta_{\{j\} \cup I} M \times P_j & \text{otherwise.} \end{array} \right\} \tag{1.1}$$

Here we denote

$$\beta_I M = \beta_{i_1} \circ \left(\beta_{i_2} \circ \left(\cdots \circ \beta_{i_q} M\right) \cdots\right),$$

if $I = \{i_1, \ldots, i_q\} \subset \{1, \ldots, k\}$.

The pair (M, f) is a *singular Σ_k-manifold of the space pair* (X, Y) , if M is a Σ_k-manifold, and

$$f : (M, \delta M) \longrightarrow (X, Y)$$

is such a map that for every index subset $I = \{i_1, \ldots, i_q\} \subset \{1, \ldots, k\}$ the map $f|_{\partial_I M}$ has the following decomposition:

$$f|_{\partial_I M} = f_I \circ pr \circ \phi_I.$$

Here the map

$$pr : \beta_I M \times P^I \longrightarrow \beta_I M$$

is the projection on the direct factor as above and the map

$$f : \beta_I M \longrightarrow X$$

is a continuous one.

Note 1.1.2 *The map f may be also decomposed: $f = f_\Sigma \circ \pi$; here $\pi : M \longrightarrow M_\Sigma$ is the projection, $f_\Sigma : M_\Sigma \longrightarrow X$ is a continuous map. Let us notice that singular Σ_k-manifolds of the point coincide with their topological models.* □

So the bordism theory $MG_*^{\Sigma_k}(\cdot)$ and cobordism theory $MG_\Sigma^*(\cdot)$ of Σ_k-manifolds are well defined. The theories $MG_*^\Sigma(\cdot)$ and $MG_\Sigma^*(\cdot)$ are determined as direct limits of the theories $MG_*^{\Sigma_k}(\cdot)$ and $MG_{\Sigma_k}^*(\cdot)$ respectively.

Theorem 1.1.2 *The theories $MG_*^\Sigma(\cdot)$ and $MG_\Sigma^*(\cdot)$ are extraordinary homology and cohomology theories respectively.*

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Proof may be given in a standard manner; it is sufficient to verify that the theories $MG^{\Sigma}_{*}(\cdot)$ and $MG^{*}_{\Sigma}(\cdot)$ satisfy the Eilenberg-Steenrod axioms; see [11]. □

Below we will deal mainly with the bordism theories; all the constructions here have a simple geometric interpretation.

Every ordinary manifold may be considered as a Σ_k-manifold with empty set of singularities and the Σ_k-manifold may be considered as a Σ_m-manifold for $m \geq k$. So the following natural transformations are well defined:

$$\pi^0_k : MG_*(\cdot) \longrightarrow MG^{\Sigma_k}_*(\cdot),$$
$$\pi^k_n : MG^{\Sigma_k}_*(\cdot) \longrightarrow MG^{\Sigma_n}_*(\cdot).$$

The operator $M \longrightarrow \beta_k M$ generates the transformation of degree $-(\dim P_k + 1)$

$$\delta_k : MG^{\Sigma_k}_*(\cdot) \longrightarrow MG^{\Sigma_{k-1}}_*(\cdot).$$

(Note that every manifold $\beta_k M$ is a Σ_{k-1}-manifold by definition.)

The transformations π^{k-1}_k, δ_k connect the theories $MG^{\Sigma_k}_*(\cdot)$ and $MG^{\Sigma_{k-1}}_*(\cdot)$ into the following exact Bockstein-Sullivan triangle:

$$\begin{array}{ccc} MG^{\Sigma_{k-1}}_* & \xleftarrow{\cdot[P_k]} & MG^{\Sigma_{k-1}}_* \\ & {\scriptstyle \pi^{k-1}_k} \searrow \qquad \nearrow {\scriptstyle \delta_k} & \\ & MG^{\Sigma_k}_* & \end{array} \tag{1.2}$$

Here we denote the transformation which is generated by direct product (from the right) on the manifold P_k by $\cdot[P_k]$.

1.2 Generalized Bockstein-Sullivan triangle

Now we construct the exact triangle which connects the theories $MG_*(\cdot)$ and $MG^{\Sigma}_*(\cdot)$ for every locally-finite sequence $\Sigma = (P_1, \ldots, P_n, \ldots)$ of closed manifolds. For this we would define a new bordism theory $MG_*^{\Sigma\Gamma(1)}(\cdot)$ which is closely connected with the theories $MG_*^{\Sigma}(\cdot)$ and $MG_*(\cdot)$.

Definition 1.2.1 *The manifold M is called a $\Sigma_k\Gamma(1)$-manifold if there are given*

(i) the partition of the manifolds

$$M = M_1 \cup \cdots \cup M_k, \quad \partial M = \delta M_1 \cup \cdots \cup \delta M_k$$

into a union of manifolds glued along boundaries, i.e. the intersection

$$M_I = M_{i_1} \cap \cdots \cap M_{i_q}$$

is a manifold for every index subset $I = \{i_1, \ldots, i_q\} \subset \{1, \ldots, k\}$ and its boundary is equal to

$$\partial(M_I) = (M_I \cap \partial M) \cup \left(\bigcup_{j \notin I} (M_I \cap M_j) \right);$$

(ii) the compatible product structures

$$\Psi_I : \gamma_I M \longrightarrow \gamma_I M \times P^I,$$

where $\gamma_I M$ are manifolds, $I = \{i_1, \ldots, i_q\} \subset \{1, \ldots, k\}$; compatibility means that the map

$$\Psi_I \circ \iota \circ \Psi_J^{-1} : \gamma_I M \times P^J \longrightarrow \gamma_I M \times P^I$$

is the identity map on the direct factor P^J for every $I \subset J$, where

$$\iota : M_I \longrightarrow M_J$$

is the corresponding inclusion. □

It is evident that a $\Sigma_k\Gamma(1)$-manifold simulates the structure which the part of the boundary of the Σ_k-manifold M,

$$\partial\widetilde{M} = \partial_1 M \cup \ldots \cup \partial_k M$$

has. Its ordinary boundary ∂M is the *boundary* ∂M *of the* $\Sigma\Gamma_k(1)$-*manifold* M; it has this structure by the definition. It should be noted that the manifolds $\gamma_I M$ as well as the manifolds $\beta_I M$ are Σ_k-manifolds.

The map

$$F : (M, \partial M) \longrightarrow (X, Y)$$

is called *the singular* $\Sigma\Gamma_k(1)$-*manifold of the space pair* (X, Y) where M is a $\Sigma\Gamma_k(1)$-manifold, $M = M_1 \cup \ldots \cup M_k$, such that the map $F|_{M_I}$ is decomposed as follows:

$$F|_{M_I} = f_I \circ pr \circ \psi_I$$

for every index subset $I = \{i_1, \ldots, i_q\} \subset \{1, \ldots, k\}$. Here

$$pr : \gamma_I M \times P^I \longrightarrow \gamma_I M$$

is the projection on the direct factor as above, and the map

$$f : \beta_I M \longrightarrow X$$

is a continuous map.

So the bordism theory $MG_*^{\Sigma\Gamma_k(1)}(\cdot)$ is well defined. We define the bordism theory $MG_*^{\Sigma\Gamma(1)}(\cdot)$ as a direct limit of the theories $MG_*^{\Sigma\Gamma_k(1)}(\cdot)$.

Consider the transformation

$$\gamma(1) : MG_*^{\Sigma\Gamma(1)}(\cdot) \longrightarrow MG_*(\cdot),$$

forgetting the $\Sigma\Gamma(1)$-structure and the transformation

$$\partial(1) : MG_*^{\Sigma}(\cdot) \longrightarrow MG_*^{\Sigma\Gamma(1)}(\cdot),$$

defined by the formula:

$$\partial(1) : [(M, f)]_\Sigma \longrightarrow \left[\left(\tilde{\partial} M, f|_{\tilde{\partial} M}\right)\right]_{\Sigma\Gamma(1)},$$

where $\tilde{\partial} M = \partial_1 M \cup \ldots \cup \partial_k M$.

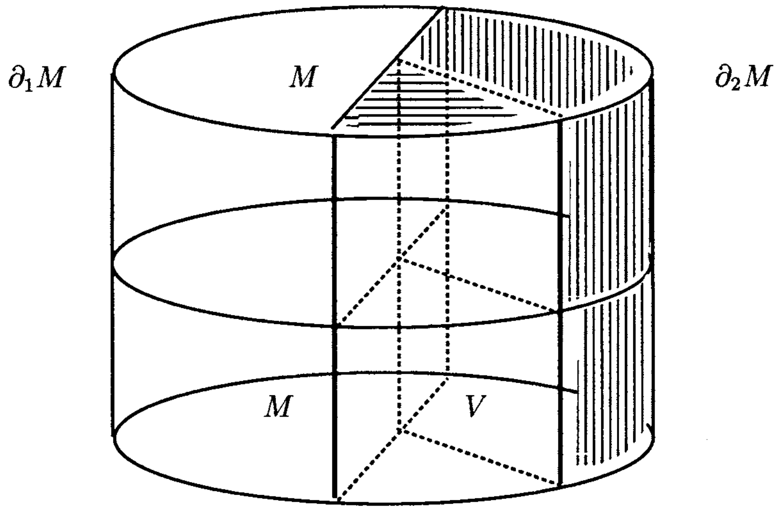
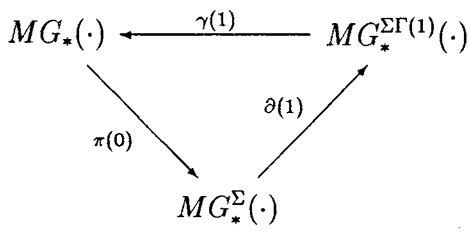


Figure 1.1: $[(L,g)] \in \mathrm{Im}\ \partial(1)$

Theorem 1.2.2 *The following triangle of the theories and transformations is exact:*



(1.3)

Proof. Let us apply the triangle (1.3) to a space X; we assume here $Y = \emptyset$ for simplicity.

1. EXACTNESS OF THE VERTEX $MG_*^{\Sigma}(\cdot)$.

The inclusion $\mathrm{Im}\ \pi \subset \mathrm{Ker}\ \partial(1)$ is obvious. Let (M,f) be a closed singular Σ-manifold, $\partial M = \partial_1 M \cup \cdots \cup \partial_k M$, such that $[(M,f)]_\Sigma \in \mathrm{Ker}\ \partial(1)$. Then there exists a singular Σ-manifold (V,G), such that

$$\partial V = \partial M, \quad G|_{\partial M} = f|_{\partial M}.$$

Consider the cylinder (see Figure 1.1)

$$((M \cup -V) \times I, (f \cup -G) \times Id) = (W, H)$$

as a Σ-bordism between the Σ-manifold

$$(M \times \{1\}, f \times \{1\}) = (M, f)$$

and the ordinary manifold

$$((M \cup -V) \times \{0\}, (f \cup -G) \times \{0\}).$$

We obtain that $[(M, f)]_{\Sigma} \in \mathrm{Im}\ \pi$.

2. Exactness of the vertex $MG_*^{\Sigma\Gamma(1)}(\cdot)$.

If (L, g) is a singular $\Sigma\Gamma(1)$-manifold and $[(L, g)] \in \mathrm{Im}\ \partial(1)$, then there exists a singular Σ-manifold (W, H), such that $\partial W = L$, $H|_{\partial V} = g$; see Figure 1.2. We obtain that $\gamma(1)([(L, g)]) = 0$ by considering (W, H) as an ordinary manifold, i.e. $\mathrm{Im}\ \partial(1) \subset \mathrm{Ker}\ \gamma(1)$. The inverse inclusion is obvious.

3. Exactness of the vertex $MG_*(\cdot)$.

Let (M, f) be an ordinary singular manifold, $[(M, f)] \in \mathrm{Ker}\ \pi$; then there exists a singular Σ-manifold (V, G) with the boundary $\partial_0 V = M$, $G|_{\partial_0 V} = f$. We have

$$(\partial_1 V \cup \cdots \cup \partial_k V) \cap \partial_0 V = \emptyset$$

because M is a closed manifold. So (V, G) may be considered as a bordism between (M, f) and the following $\Sigma\Gamma(1)$-manifold

$$(\partial_1 V \cup \cdots \cup \partial_k V, G|_{\partial_1 V \cup \cdots \cup \partial_k V}).$$

The inclusion $\mathrm{Im}\ \gamma(1) \subseteq \mathrm{Ker}\ \pi$ is obvious. □

We have the commutative diagram for every space pair (X, Y)

$$\begin{array}{ccccc} \xrightarrow{\delta} MG_*(X,Y) & \xrightarrow{\times[P]} & MG_*(X,Y) & \xrightarrow{\pi} & MG_*^{\Sigma}(X,Y) \xrightarrow{\delta} \\ \downarrow \mathrm{Id} & & \downarrow \omega & & \downarrow \mathrm{Id} \\ \xrightarrow{\delta} MG_*(X,Y) & \xrightarrow{\gamma(1)} & MG_*^{\Sigma\Gamma(1)}(X,Y) & \xrightarrow{\pi(0)} & MG_*^{\Sigma}(X,Y) \xrightarrow{\delta} \end{array} \qquad (1.4)$$