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Stability, Instability and Chaos:

an introduction to the theory of nonlinear differential equations

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Preface

As the theory of dynamical systems and chaos develops, more and more recent results are filtering through to undergraduate courses. The aim of this book is to provide a coherent account of the qualitative theory of ordinary differential equations which deals in an even handed and consistent way with both the 'classical' results of Poincaré and Liapounov and the more recent advances in bifurcation theory and chaos. The book covers two undergraduate courses: a first course in nonlinear differential equations and an introduction to bifurcation theory.

Throughout, the emphasis is on understanding and the ability to apply theory to examples rather than on rigorous mathematical developments. Although there are theorems, the level of rigour is not that of a pure mathematical text. None the less, it is vital to appreciate the restrictions and limitations of any method and so wherever possible I have stated results in a precise form. The choice of topics has also been influenced by a desire to cover material which can be examined sensibly.

This book has developed out of courses given to third year undergraduates at the University of Warwick and the University of Cambridge. In both places I have been fortunate to inherit the notes of previous lecturers: Tony Pritchard at Warwick, Peter Swinnerton-Dyer, John Hinch and others at Cambridge. The first seven chapters owe an enormous debt to these people and in some places it is hard for me to see where they end and I begin (although I retain full responsibility for any errors). There are a number of exercises, some in the main text and some at the end of chapters. Those in the main text are intended to help the reader follow some argument. There is some repetition of important exercises and some exercises are answered later in the text. Many of the exercises at the end of the chapters are taken from example sheets and examination questions from the third year courses on *Nonlinear Differential Equations* and *Dynamical Systems* at the University of Cambridge.

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James Glover, Alistair Mees and a class at the University of Western Australia in Perth acted as guinea pigs for an early version of the book. Their comments and corrections were immensely useful. An anonymous reader for CUP also made a number of good suggestions, and I would like to thank Alan Harvey, Roger Astley and David Tranah at CUP for their support. The diagrams were expertly produced by Margaret Downing (by hand) and Alastair Rucklidge (by computer). I have received help and encouragement from many people whilst writing this book. In particular, Bob Devaney, Toby Hall, Guilermo Procida, Mike Proctor, Colin Sparrow, Ian Stewart, Sebastian van Strien and Nigel Weiss all made helpful remarks or pointed out painful mistakes, and Fiona Russell helped me keep a sense of perspective and focus throughout the period of writing.

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Notation

Most of the notation used in this book is standard and either does not need explanation or is explained in the text. Vectors and vector valued functions (vector fields) are neither underlined nor in bold. Derivatives with respect to time, t, which is generally the independent variable, are denoted by dots (as in $\dot{x}$) whilst derivatives of real valued functions of a single variable which is not time are denoted by primes (as in $f'(x)$). Partial derivatives are sometimes indicated by subscripts, so f_{xy} would denote $\frac{\partial^2 f}{\partial x \partial y}$. Derivatives of vector fields are viewed as matrices, so if $f : \mathbf{R}^n \to \mathbf{R}^n$ is a smooth vector field which is a function of $x \in \mathbf{R}^n$, then the derivative (or Jacobian matrix) is the $n \times n$ matrix $Df(x)$ defined in Cartesian coordinates, $f(x) = (f_1(x), \ldots, f_n(x))$, $x = (x_1, \ldots, x_n)$, by

$$[Df(x)]_{ij} = \frac{\partial f_i}{\partial x_j}(x),$$

$0 \leq i, j \leq n$. If A is an open set then the closure of A is denoted by $cl(A)$, whilst if B is a closed set then the interior of B is denoted by $int(B)$. The function $sign(x)$ is simply the sign of x, i.e. $+1$ if x is positive, -1 if x is negative and 0 if x equals zero.