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Introduction

Differential equations are used throughout the sciences to model dynamic processes. They provide the most simple models of any phenomenon in which one or more variables depend continuously on time without any random influences. They are also fascinating mathematical objects in their own right. If a differential equation is derived from some physical situation it is clearly desirable to know something about solutions to the equation. Indeed, there is little point deriving a model if it is then impossible to gain any information from it! This poses a big problem. Whilst most differential equations in university courses have closed form solutions, typical nonlinear differential equations do not have solutions which can be written down in terms of familiar special functions such as sines and cosines. This means that when faced with general (nonlinear) differential equations we need to change our approach. We will rarely solve differential equations, instead we will try to obtain qualitative information about the long term, or *asymptotic*, behaviour of solutions: are they periodic? eventually periodic? attracting? and so on. This shift from the quantitative to the qualitative is reflected in a shift in the mathematical techniques which are used to analyse equations: much of the analysis will be geometric rather than analytic.

Initially we will concentrate on *hyperbolic* solutions. Roughly speaking a solution is hyperbolic if all sufficiently small perturbations of the defining differential equation have similar behaviour close to that solution (this is not simply a statement about continuity). This leads on to the idea of differential equations which depend on a parameter. For example, if the differential equation models some physical situation then a coefficient in the equation may depend upon temperature. In such a case it may be useful to know the dependence of solutions on the ambient temperature, i.e. to analyse the differential equation for several different values of the coefficient. If at some value of this coefficient a solution is hyperbolic, then a small change in the coefficient, and hence a small change in the ambient temperature, does not alter the qualitative behaviour of the system *near that solution*. The second half of this book introduces ideas from bifurcation theory, which describes the qualitative changes that can occur near non-hyperbolic solutions. This corresponds to situations in which small changes in the coefficients of the defining equations can lead to qualitatively different behaviour of solutions.

Before these terms are given more precise definitions it is worth thinking about the possible behaviour we might expect to meet. A standard introductory example in physics is the model of the ideal pendulum (Fig. 1.1). A simple application of Newton's laws of motion shows that the angle θ of the pendulum changes with time satisfying an equation of the form

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0 \tag{1.1}$$

after rescaling time so as to make the various physical constants that appear equal to unity.

If we consider only small amplitude oscillations then $\sin\theta \approx \theta$ and so we obtain the simplified equation

$$\frac{d^2\theta}{dt^2} + \theta = 0 \tag{1.2}$$

which has solutions $\theta = A\sin t + B\cos t$, where A and B are constants determined by the initial position and angular velocity of the pendulum. If $A = B = 0$ then $\theta = 0$ is a solution. This solution corresponds to the

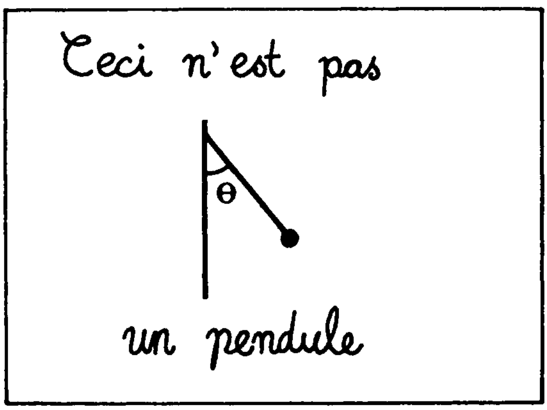

Fig. 1.1 The ideal pendulum (with apologies to Magritte).

stationary pendulum, where it simply hangs directly downwards with no oscillation. There is no motion. So, the first sort of dynamics that we can identify is trivial: no motion. However, for more general choices of initial conditions the solution $A\sin t + B\cos t$ is periodic: the position and angular velocity of the solution is the same at time t and time $t+2\pi$. This is called periodic motion with period 2π and corresponds to the simple periodic oscillations of the pendulum. By complicating the equation a little we can get examples of more complicated dynamics. For example, if

$$\frac{d^2\theta}{dt^2} + \theta = a(1-\omega^2)\cos\omega t, \qquad (1.3)$$

with $\omega \neq \pm 1$, then solutions are

$$\theta(t) = A\sin t + B\cos t + a\cos\omega t. \qquad (1.4)$$

Is this solution periodic? The first two terms are periodic with period 2π and the third term is periodic with period $\frac{2\pi}{\omega}$. The solution is periodic if there exists a time T such that both $\theta(0) = \theta(T)$ and $\frac{d\theta}{dt}(0) = \frac{d\theta}{dt}(T)$, i.e. if T is a multiple of both 2π and $\frac{2\pi}{\omega}$. So solutions are periodic if there exist integers p and q such that $2\pi q = \frac{2\pi p}{\omega}$ or $\omega = \frac{p}{q}$, a rational number. If ω is irrational then we say that the solution is *quasi-periodic* with two independent frequencies. Although the solution is not periodic it does have a regular structure (see Fig. 1.2).

One of the most exciting developments in the recent theory of differential equations is the discovery that relatively simple differential equations can have solutions which are much more complicated than these periodic and quasi-periodic solutions. Very roughly, a differential equation is said to be *chaotic* if there are bounded solutions which are neither periodic nor quasi-periodic and which diverge from each other locally. The existence of chaotic solutions has had a profound effect on thinking in many disciplines. One immediate corollary of the local divergence of nearby solutions is that one loses predictive power in practical situations. The solutions of differential equations are deterministic in the sense that if the initial conditions are precisely specified then the solution is completely determined and so, in principle, we should be able to predict the value of the solution at some later time. Of course, in practice the initial condition can only be known to some finite precision and so if the equation is chaotic we rapidly lose information about the system since our solution through the approximate initial condition does not stay close to the desired solution. In experiments this can manifest itself in an apparent unrepeatability of the results: many physicists have

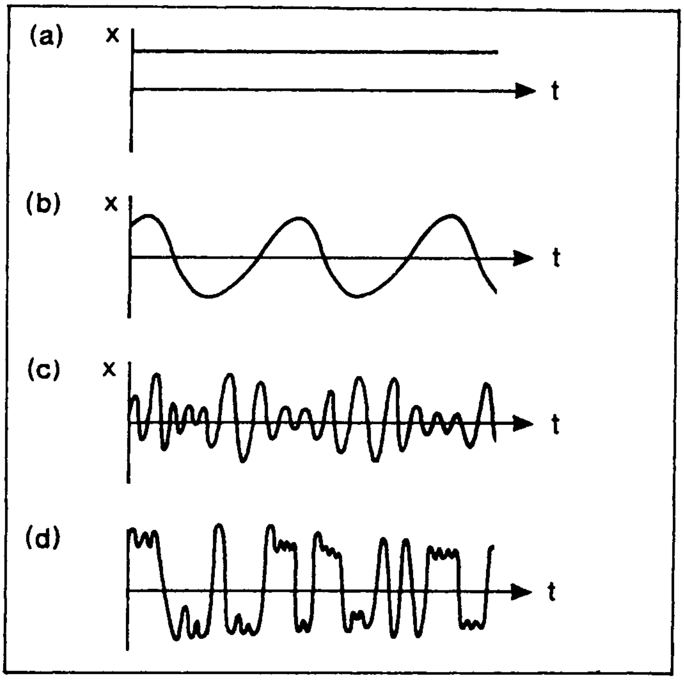

Fig. 1.2 Time series for differential equations. (a) No motion; (b) periodic motion; (c) quasi-periodic motion; (d) possibly chaotic motion.

dusted off experiments which were rejected in the 1960s on the grounds that results were not repeatable. At the time it was assumed that there was some sort of background noise or random fluctuation which had not been eliminated, but these are now recognised as being examples of chaotic behaviour. The results *are* repeatable, but only if looked at from the right point of view. A useful example is the pinball machine: imagine trying to reproduce a sequence of scores! Yet there is no random element, it is all just Newton's laws in action (assuming that the table is not being jiggled overvigorously). The source of this complexity lies in the rounded buffers: small differences in the trajectory of the ball are magnified each time the ball strikes a buffer. Another, more mathematical, example is the difference equation

$$x_{n+1} = 10x_n \pmod 1. \tag{1.5}$$

Given an initial number, x_0, with decimal expansion $0.a_0a_1a_2\ldots$, this difference equation generates a new number $x_1 = 0.a_1a_2a_3\ldots$ which generates a new number x_2 and so on. Now consider the sequence $(x_i)_{i\geq 0}$.

We say the sequence is eventually periodic of period p if $x_n = x_{n+p}$ for all $n \geq N$. Since x_n is obtained from x_{n-1} by simply deleting the first term in the decimal expansion of the number we can see that every rational number is eventually periodic, since the decimal expansion of a rational number is eventually periodic, and every irrational number is aperiodic. Furthermore, suppose you wanted to predict the motion of a given point on the interval but the number is only known to a finite precision (7 decimal places, say). Then you would know x_1 to six decimal places, x_2 to five decimal places, x_3 to four decimal places, x_4 to three decimal places ... and x_7 could be anywhere! This illustrates the loss in predictive power which also seems to be at work in weather forecasting and many other situations.

We will now begin to describe the framework which will be the basis of this book. A differential equation is an equation of the form

$$\frac{d^n x}{dt^n} = F\left(t, x, \frac{dx}{dt}, \dots, \frac{d^{n-1}x}{dt^{n-1}}\right) \tag{1.6}$$

and (modulo some technical assumptions described in Section 1.2) these equations have solutions given some set of initial conditions at $t = t_0$

$$x(t_0) = c_1, \ \frac{dx}{dt}(t_0) = c_2, \dots, \frac{d^{n-1}x}{dt^{n-1}}(t_0) = c_n. \tag{1.7}$$

Throughout this book we shall choose to consider differential equations in the form

$$\dot{y} = f(y,t), \quad y \in \mathbf{R}^n, \ f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n, \tag{1.8}$$

where the dot denotes differentiation with respect to time. We shall not be overly concerned about optimal smoothness conditions on f for results to hold, but will assume that f is sufficiently smooth for the Taylor expansions and other techniques used to be valid. Note that any equation of the form (1.6) can be rewritten as (1.8) by setting

$$\frac{d^k x}{dt^k} = y_{k+1} \tag{1.9}$$

for $0 \leq k \leq n-1$, in which case

$$\dot{y}_k = y_{k+1}, \quad 1 \leq k \leq n-1 \tag{1.10a}$$

$$\dot{y}_n = F(t, y_1, \dots, y_n) \tag{1.10b}$$

i.e. if $y = (y_1, y_2, \dots, y_n)$ and

$$f(y,t) = (y_2, \dots, y_n, F(t, y_1, \dots, y_n))$$

then $\dot{y} = f(y, t)$. The number n is called the order of the differential equation, and an n^{th} order differential equation needs n initial conditions to specify a solution. These can be thought of as the n constants which arise in the n integrations required to solve the equation.

A particularly simple example, which can be solved in general, is the linear differential equation

$$\dot{y} = Ay \tag{1.11}$$

where A is an $n \times n$ matrix with constant coefficients. If the initial condition at $t = 0$ is y_0 then this equation has solutions

$$y = e^{tA} y_0. \tag{1.12}$$

So provided we understand the exponential of a matrix we can solve this differential equation exactly. Unfortunately, these are about the only equations which can be solved exactly, and bitter experience has taught scientists that the world is not linear. Hence, to understand more complicated (nonlinear) models we must learn how to treat nonlinear equations which we are unable to solve. Before doing this we should think a little harder about what it means to solve a differential equation.

1.1 Solving differential equations

Let's start with a simple example and see how some of the standard techniques for solving differential equations work. Consider the equation

$$\ddot{x} + x = 0 \tag{1.13}$$

with initial conditions $x(0) = a$ and $\dot{x}(0) = b$. This equation should be familiar; it is the equation for simple harmonic motion, (1.2), with solution

$$x(t) = a \cos t + b \sin t. \tag{1.14}$$

This solution is meaningless unless the properties of the functions sine and cosine of t are well known, which, of course, they are. So, how did we solve this equation? We shall sketch three different methods, at least one of which is, I hope, familiar to you.

Method 1

Note that if we set $y = \dot{x}$ then (1.13) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

with initial conditions $x(0) = a$ and $y(0) = b$. In matrix notation with $w = (x, y)^T$ and the matrix on the righthand side of this equation denoted by A, this becomes $\dot{w} = Aw$, with $w(0) = (a, b)^T$. As pointed out in the preamble to this chapter, in equation (1.12), this has solutions $\exp(tA)w(0)$ and so we need to calculate the matrix $\exp(tA)$, which we do by means of the series definition

$$\exp(tA) = \sum_{k=0}^{\infty} \frac{t^n A^n}{n!}.$$

It is a simple exercise to show that

$$A^{2n} = \begin{pmatrix} (-1)^n & 0 \\ 0 & (-1)^n \end{pmatrix} \text{ and } A^{2n+1} = \begin{pmatrix} 0 & (-1)^n \\ (-1)^{n+1} & 0 \end{pmatrix}$$

and so

$$\exp(tA) = \begin{pmatrix} \sum_n t^{2n}(-1)^n/(2n)! & \sum_n t^{2n+1}(-1)^n/(2n+1)! \\ \sum_n t^{2n+1}(-1)^{n+1}/(2n+1)! & \sum_n t^{2n}(-1)^n/(2n)! \end{pmatrix},$$

which we recognise as being series solutions for sine and cosine of t to give

$$\exp(tA) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Hence $x(t) = a\cos t + b\sin t$ and $y(t) = -a\sin t + b\cos t$.

Method 2

Try a trial solution of the form $x = e^{ct}$ and solve for c. Then note that since (1.13) is linear, if x_1 and x_2 are independent solutions then the general solution is a sum of x_1 and x_2. Substituting $x = e^{ct}$ into the differential equation gives $(c^2 + 1)e^{ct} = 0$ so $c = \pm i$. The solution is therefore $x(t) = c_1 e^{it} + c_2 e^{-it}$ where the complex coefficients c_1 and c_2 are determined from the initial conditions.

Method 3

Note that the differential equation can be written as

$$\left(\frac{d}{dt}+i\right)\left(\frac{d}{dt}-i\right)x=0.$$

Set $v(t)=(\frac{d}{dt}-i)x$ so

$$\left(\frac{d}{dt}+i\right)v=0 \text{ or } e^{-it}\frac{d}{dt}(ve^{it})=0.$$

Hence $ve^{it}=c_1$, or $v=c_1e^{-it}$. Now replace v by the definition of v in terms of x and solve another linear first order differential equation to obtain x as a sum of $e^{\pm it}$ as in method 2.

In all three methods we have assumed and used properties of the exponential function and of sine and cosine in order to solve integrals or guess solutions. But what happens if the differential equation is more complicated and, in particular, if it is nonlinear? As an example consider

$$\ddot{x}+x-x^3=0. \tag{1.15}$$

It does not take much effort to see that none of the methods described above can be applied to this equation; in all cases we are either unable to start or end up with integrals that we cannot solve. Nonetheless these equations do have solutions (snoidal functions, which are defined in terms of elliptic integrals). So, if you knew about elliptic functions you could solve the differential equation (i.e. write down the solution in terms of these functions). To what extent is this useful? Old fashioned books of mathematical functions will often have elliptic functions in tabulated form, so in principle it would be possible to find the solution at a given time approximately using these. There are also formulae for these functions which are valid in particular regimes. However, because we are not familiar with these functions the closed form solution is not, on its own, very helpful.

As a further example consider

$$\ddot{x}+x+x^3+x^7=0. \tag{1.16}$$

Once again, these have solutions given initial values of x and $\dot{x}$ although to the best of my knowledge they are not tabulated anywhere. So, instead of writing this book I could define the solutions of these equations to be the functions $Gl(t)$ (Gl for Glendinning, perhaps) and write a book exploring the properties of these functions and giving tabulated approximations. I fear that such a book would bring me neither fame nor fortune. The problem is that the functions would not have sufficiently wide application to be interesting and, as we shall see, many properties of solutions can be deduced without resorting to the tiresome exercise of solving the differential equation either numerically or in certain limits to obtain approximations to exact solutions.

Let us pause to take stock for a minute. I hope that these examples illustrate the point that closed form solutions are not always possible to find, and that even when they can be found they may not be particularly useful. This suggests that we need an alternative way of looking at the solutions of differential equations. To develop this possibility we need to be clear about what we consider to be the truly important feature of solutions. Perhaps the most important feature of the linear differential equation $\ddot{x}+x=0$ is that all solutions are periodic; that is, they repeat themselves after each period of 2π (we can see this from the 2π periodicity of the functions sine and cosine). There is one special case. If the initial condition is $(a,b)=(0,0)$ then $x=0$ for all time. Hence the qualitatively useful information which we deduce from the exact solutions is that if x and $\dot{x}$ are initially both zero then they remain zero for all time (this is called a stationary point) whilst otherwise the solutions are periodic. Since we don't know enough about elliptic functions or Glendinning functions we cannot say what we might consider to be important for the other two examples. It may seem reasonable to look at the asymptotic behaviour, i.e. what happens as $t\to\infty$ and see whether we can understand that motion in some way. For example, if a solution to some equation gives $x(t)=3/(2+x_0e^{-t})$ then, as $t\to\infty$, x tends to the constant value $3/2$, and the way in which it approaches this constant value is, for many purposes, less important than the fact that for large enough values of t the solution is arbitrarily close to $3/2$. Thus we might transfer attention from the exact solution to having a general picture of the type of behaviour observed after some time has elapsed.

Another way of looking at the equations described above is to multiply through by $\dot{x}$ and note that (by the chain rule)

$$\frac{d}{dt}(\tfrac{1}{2}\dot{x}^2)=\dot{x}\ddot{x} \quad\text{and}\quad \frac{d}{dt}(\tfrac{1}{n}x^n)=\dot{x}x^{n-1}.$$

Hence for the linear equation of simple harmonic motion we find

$$\dot{x}(\ddot{x}+x)=0=\frac{d}{dt}(\tfrac{1}{2}\dot{x}^2+\tfrac{1}{2}x^2). \tag{1.17}$$

Therefore we can integrate once to obtain

$$\tfrac{1}{2}\dot{x}^2+\tfrac{1}{2}x^2=C \tag{1.18}$$

for some positive constant C which depends upon the initial conditions. This implies that solutions plotted in the $(x, \dot{x})$ plane lie on concentric circles centred at the origin as shown in Figure 1.3a, which is called a phase portrait or phase space diagram of the system. These concentric circles represent the periodic solutions, and the arrows on the curves indicate the direction of time. The arrows of time can easily be deduced by noting that the equation can be written as $\dot{x} = y$, $\dot{y} = -x$ and so $\dot{x} > 0$ in $y > 0$ and $\dot{x} < 0$ in $y < 0$. Hence the x coordinate increases on solutions when $y > 0$, and x decreases when $y < 0$. In this description then, we have lost the precise parametrization by time, but we retain the important geometric information that solutions lie on closed curves (periodic orbits) unless $C = 0$, which gives the single point at the origin. Note that we could go on (Method 4) to solve for the solution explicitly by solving

$$\frac{dx}{dt} = \sqrt{2C - x^2}, \quad C \geq 0,$$

that is,

$$\int_{x_0}^{x(t)} \frac{dz}{\sqrt{2C - z^2}} = \int_0^t d\tau, \tag{1.19}$$

which can be solved without a great deal of sophistication. However, this final step is, at least to some extent, unnecessary, since we have already been able to deduce the important features of the solutions from the structure of solution curves in the $(x, \dot{x})$ plane.

The function $\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2$ is called a first integral of the problem, and it is (again, unfortunately) rare to be able to obtain first integrals with such ease. However, both the nonlinear examples of this section can be approached in this way. Multiplying $\ddot{x} + x - x^3 = 0$ by $\dot{x}$ gives

$$\dot{x}(\ddot{x} + x - x^3) = 0 = \frac{d}{dt}\left(\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}x^2 - \tfrac{1}{4}x^4\right)$$

and so solutions lie on the curves

$$\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}x^2 - \tfrac{1}{4}x^4 = C \tag{1.20}$$

in the $(x, \dot{x})$ plane. These are sketched in Figure 1.3b, from which we see immediately that there is a bounded family of periodic solutions about the origin, and a family of unbounded solutions. We shall describe the limiting solutions which separate these two families of solutions later on. If we wanted to give the complete solution of these equations we would