An Evolution Equation for the Intersection Local Times of Superprocesses[1]

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1 Introduction

The primary aim of this paper is to establish evolution equations for the *intersection local time* (ILT) of the super Brownian motion and certain super stable processes. We shall proceed by carefully defining the requisite concepts and giving all of our main results in the Introduction, while leaving the proofs for later sections. The Introduction itself is divided into four sections, which treat, in turn, the definition of the superprocesses that will interest us, the definition of ILT and some previous results, our main result – a Tanaka-like evolution equation for ILT – and an Itô formula for measure-valued processes along with a description of how to use it to derive the evolution equation. Some technical lemmas make up Section 2 of the paper, while Section 3 is devoted to proofs.

In order to conserve space, we shall motivate neither the study of superprocesses *per se* – other than to note that they arise as infinite density limits of infinitely rapidly branching stochastic processes – nor the study of ILT – other than to note that this seems to be important for the introduction of an intrinsic dependence structure for the spatial part of a superprocess. Good motivational and background material on superprocesses can be found in Dawson (1978, 1986), Dawson, Iscoe and Perkins (1989), Ethier and Kurtz (1986), Roelly-Coppoletta (1986), Walsh (1986) and Watanabe (1968), as well as other papers in this volume. Material on ILT can be found in Adler, Feldman and Lewin (1991), Adler and Lewin (1991), Adler and Rosen (1991), Dynkin (1988) and Perkins (1988).

[1]Research supported in part by US-Israel Binational Science Foundation (89-298), Air Force Office of Scientific Research (AFOSR 89-0261) and the Israel Academy of Sciences (702-90).

(a) Super Brownian Motion and Super Stable Processes. We require some notation.

$$M = M(\Re^d) = \{\mu : \mu \text{ is a Radon measure on } \Re^d\}.$$

$$M_q = M_q(\Re^d) = \{\mu : \mu \in M, \ \textstyle\int_{\Re^d}(1+\|x\|)^{-q}\mu(dx) < \infty\}.$$

$$C_o = C_o(\Re^d) = \{f : \Re^d \to \Re\ ,\ f \text{ continuous, } \lim_{\|x\|\to\infty} f(x) = 0\}.$$

$$C_K = C_K(\Re^d) = \{f : \Re^d \to \Re\ ,\ f \text{ continuous with compact support}\}.$$

The Sobolev space of functions whose k-th derivatives are in $\mathcal{L}^p$ is denoted by $W^{k,p}$. The d-dimensional Laplacian is denoted by Δ, and the fractional Laplacian by $\Delta_\alpha = -(-\Delta)^{\alpha/2}$, $\alpha \in (0,2)$. (c.f. Yosida (1965).) With some abuse of notation, we shall let $\Delta_2 \equiv \Delta$. The domain of an operator A is denoted by $D(A)$, $\mathcal{S}_d$ is the Schwartz space of rapidly decreasing functions on $\Re^d$, and $\mathcal{S}'_d$ is the space of tempered distributions on $\Re^d$. $(\Omega,\ \mathcal{F},\ \mathcal{F}_t,\ P)$ is a filtered probability space.

The *super Brownian motion*, starting at $\mu \in M_q$, is a M_q-valued, $\mathcal{F}_t$-adapted, strong Markov process with $X_0 = \mu$ and satisfying

$$\begin{aligned} \langle \varphi, X_t \rangle &= X_t(\varphi) \\ &= \mu(\varphi) + Z_t(\varphi) + \int_0^t X_s(\Delta\varphi)\,ds, \end{aligned} \tag{1.1}$$

for every $\varphi \in D(\Delta) \cap \mathcal{S}_d$, where Z_t is a continuous $\mathcal{F}_t$ martingale measure with increasing process

$$[Z(\varphi)]_t = \int_0^t X_s(\varphi^2)ds. \tag{1.2}$$

The *super* (symmetric) *stable process* is defined via the same recipe, but with Δ_α replacing Δ and Z^α replacing Z throughout.

For the remainder of this paper, we shall take $X_0 = \mu =$ Lebesgue measure, so we shall implicitly assume that $q > d$ throughout.

(b) Intersection Local Time (ILT). At a heuristic level, the (self) intersection local time of a measure-valued process is a set indexed functional of the form

$$L(B) = \int_B dsdt \int_{\Re^d\times\Re^d} \delta(x-y)\,X_s(dx)X_t(dy), \tag{1.3}$$

where B is a finite rectangle in $[0,\infty) \times [0,\infty)$ and δ is the Dirac delta function. A more precise definition of $L(B)$ is obtained by replacing the delta function by an approximate delta function $f_\epsilon \in \mathcal{S}_d$, and then taking an $\mathcal{L}^2$ limit as $\epsilon \to 0$. In particular, let $\{f_\epsilon\}_{\epsilon>0}$ be a collection of positive C^∞ functions with $\int f_\epsilon(x)dx = 1$ for all ϵ, such that $f_\epsilon \to \delta$ as $\epsilon \to 0$, where convergence is in the sense of distributions.

Replace (1.3) by

$$L_\epsilon(B) = \int_B dsdt \int_{\Re^d \times \Re^d} f_\epsilon(x-y) X_s(dx) X_t(dy). \tag{1.4}$$

There are two qualitatively different cases that arise in this formulation. The first, and by far the simpler case, arises when the set B does not intersect the diagonal $D = \{(t,t) : t \geq 0\}$. This case has been considered in detail in Dynkin (1988) to whom we refer the reader for details, noting here merely the fact that Dynkin has established the existence of $L(B)$ as an $\mathcal{L}^2$ limit of $L_\epsilon(B)$ for all dimensions $d \leq 7$, and that he treats a wider class of superprocesses than that considered in this paper.

The more difficult case, in which $B \cap D \neq \emptyset$, requires a renormalization argument, since any attempt at a direct approach to (1.3) as a limit of $L_\epsilon(B)$ yields a plethora of infinities. In order to state what is known in this case, we adopt the notation

$$\begin{aligned} \langle \varphi, X_t \rangle &= \langle \varphi(x),\ X_t(dx) \rangle = \int_{\Re^d} \varphi(x) X_t(dx) \\ \langle \psi, X_s \times X_t \rangle &= \langle \psi(x,y),\ X_s(dx) X_t(dy) \rangle = \int_{\Re^d \times \Re^d} \psi(x,y)\, X_s(dx) X_t(dy). \end{aligned}$$

Then Rosen (1990) has proven the following result, which, contrary to the assumption made above, holds only if the initial measure $\mu = X_0$ is finite.

Theorem 1.1 (Rosen (1990)) . *Let $\hat{\gamma}_\epsilon(T)$ be the approximate, renormalized, intersection local time defined by*

$$\begin{aligned} \hat{\gamma}_\epsilon(T) \;=\; & \int_0^T dt \int_0^t ds\ \big(\langle f_\epsilon(x-y),\ X_s(dx)\, X_t(dy) \rangle \\ & - E\{ \langle f_\epsilon(x-y),\ X_s(dx)\, X_t(dy) \rangle | \mathcal{F}_s \} \big), \end{aligned} \tag{1.5}$$

where the particular sequence $\{f_\epsilon = \epsilon^{-d} f(\cdot/\epsilon)\}$, for some symmetric, positive $f \in C_K$ with $\int f(x)dx = 1$ is chosen for the approximate δ-functions. If X_t is a super Brownian motion and $d = 4$ or 5, then $\hat{\gamma}_\epsilon(T)$ converges in $\mathcal{L}^2$, to a limit independent of f, as $\epsilon \to 0$. $\mathcal{L}^2$ convergence also holds if X_t is a super stable process of index α and $d/3 < \alpha \leq d/2$. The limit process is denoted by $\hat{\gamma}(T)$ and is called Rosen's ILT process for X.

Rosen's results also describe what happens when the conditions on d and α are not satisfied, and cover superprocesses defined over smooth elliptic diffusions.

It is worthwhile to note at this point that, as we shall see later, the renormalization in (1.5) obtained from the conditional expectation is not unique. Rosen's choice of renormalization arises naturally from his style of proof, which is heavily based on moment arguments. Our approach, which is stochastic analysis based, will yield a slightly different renormalization. Note also that if $d \leq 3$ in the Brownian case, or $\alpha > d/2$ in the stable case, the renormalization is not necessary, as both terms in (1.5) have finite second moment for all $\epsilon > 0$. Furthermore, in both of these cases, the super process has a well-defined local time, which can be used to both define the ILT and to develop an evolution equation for it. (cf. the discussion in Adler and Rosen (1991) regarding a similar situation for the Brownian and stable density processes.)

The primary aim of this paper is to go beyond the existence results of Theorem 1.1, and derive an evolution equation describing the spatial (via a parameter to be introduced below) and temporal development of $\gamma(T)$.

(c) An Evolution Equation Let p_t and p_t^α, $\alpha \in (0,2)$ be, respectively, the transition densities of Brownian and symmetric stable processes. Thus, with a slight abuse of notation:

$$\begin{aligned} p_t^2(x,y) &= p_t(x,y) = p_t(x-y) \\ &= \tfrac{1}{(4\pi t)^{d/2}} e^{-\|x-y\|^2/4t}, \end{aligned}$$

$$\begin{aligned} p_t^\alpha(x,y) &= p_t^\alpha(x-y) \\ &= \tfrac{1}{(2\pi)^d} \int_{\Re^d} \exp(-ip\cdot(x-y) - t\|p\|^\alpha)\, dp. \end{aligned}$$

The corresponding Green's functions, for $\alpha \in (0,2]$ and $\lambda \geq 0$, are defined by

$$G_\alpha^\lambda(x-y) = G_\alpha^\lambda(x,y) = \int_0^\infty e^{-\lambda t} p_t^\alpha(x,y)\, dt \qquad (1.6)$$

where we allow ourselves the luxury of writing G_α^λ as a function of either one or two parameters at will. Note that, depending on the values of α and d, G_α^0 may be identically infinite. For all α and d, however, $G_\alpha^\lambda(x,y)$ is finite if $\lambda > 0$ and $x \neq y$. Furthermore, $G_\alpha^\lambda \in \mathcal{L}^1$ for all $\lambda > 0$. We shall list a number of properties of G_α^λ in the following section. The property of central importance to us is the fact that the distributional equation

$$(-\Delta_\alpha + \lambda)u = \delta, \qquad (1.7)$$

where δ is the Dirac delta function, is solved by $u = G_\alpha^\lambda$. That is, for every test function $\varphi \in \mathcal{S}_d$,

$$\int_{\Re^d} ((-\Delta_\alpha + \lambda)G_\alpha^\lambda)(x)\varphi(x)dx = \varphi(0).$$

The fact that $G_\alpha^\lambda \in \mathcal{L}^1$ for $\lambda > 0$ implies that G_α^λ can also be treated as an $\mathcal{S}'_d$ distribution. Thus, there exists a family $\{G_\epsilon\}_{\epsilon>0}$ of C_K functions such that $G_\epsilon \to G_\alpha^\lambda$ as $\epsilon \to 0$, in $\mathcal{S}'_d$. Since Δ_α is a continuous operator on $\mathcal{S}'_d$ (cf. Hörmander (1985), page 70) it follows that

$$G_\epsilon^{\alpha,\lambda} := (-\Delta_\alpha + \lambda)\, G_\epsilon \to \delta \tag{1.8}$$

as $\epsilon \to 0$, where convergence is again in $\mathcal{S}'_d$. Thus it is not unreasonable to attempt to define a new approximate, renormalized ILT by setting, for every $\varphi \in \mathcal{S}_d$,

$$\begin{aligned}\gamma_\epsilon^\lambda(T,\varphi) &= \int_0^T dt \int_0^t ds \langle \varphi(x) G_\epsilon^{\alpha,\lambda}(x-y),\, X_s(dx)\, X_t(dy)\rangle \\ &\quad - \int_0^T \langle \varphi(x) G_\epsilon(x-y),\, X_t(dx)\, X_t(dy)\rangle dt. \end{aligned} \tag{1.9}$$

There are a number of differences between the approximate ILT, $\gamma_\epsilon^\lambda(T,\varphi)$, and the approximate ILT $\gamma_\epsilon(T)$, of Rosen. Firstly, the addition of the test function φ gives information on the spatial dispersion of self-intersections that is not available otherwise. (The addition of this parameter is also necessitated by the fact that we work with an infinite initial measure.) Secondly, γ_ϵ^λ is not so much *an* ILT as a *family* of ILT's indexed by the parameter λ. This emphasizes the non-uniqueness of the renormalization. The main difference, however, lies in the structure of the renormalization. As the proofs below will show, the renormalizing second term of (1.9) arises naturally from stochastic analysis considerations. The fact that it involves the "doubling" of the measure X – i.e. it involves $X_s \times X_t$ only for $t = s$ – is highly reminiscent of the renormalized, self-intersection local time for $\Re^d$ valued processes for which the renormalization involves the removal of "local double points" of the process. For this reason, we find our renormalization esthetically more appealing than that of (1.5).

Theorem 1.2 *Let X_t be a super Brownian motion or super stable process, and let $\gamma_\epsilon^\lambda(T,\varphi)$ be its approximate renormalized intersection local time as defined by (1.9). If $d = 4$ or 5 in the Brownian case, or $d/3 < \alpha \le d/2$ in the* *stable case, then for all $\lambda > 0$, all $T \in (0,\infty)$ and all $\varphi \in \mathcal{S}_d$, $\gamma_\epsilon^\lambda(T,\varphi)$ converges in $\mathcal{L}^2$ to a finite limit $\gamma^\lambda(T,\varphi)$ as $\epsilon \to 0$ which we call a renormalized ILT of X_t, and which is independent of the approximating sequence $\{G_\epsilon\}$. Furthermore, γ^λ has the following representation in terms of the process X_t and the martingale measure Z_t:*

$$\begin{aligned}\gamma^\lambda(T,\varphi) &= \int_0^T \int_{\Re^d} \left\{ \int_0^t \langle G_\alpha^\lambda(x-y)\varphi(x),\, X_s(dx)\rangle\, ds \right\} Z(dt,dy) \\ &\quad + \lambda \int_0^T dt \int_0^t ds\, \langle G_\alpha^\lambda(x-y)\varphi(x),\, X_s(dx)\, X_t(dy)\rangle \\ &\quad - \int_0^T \langle G_\alpha^\lambda(x-y)\varphi(x),\, X_t(dx)\, X_T(dy)\rangle\, dt. \qquad (1.10)\end{aligned}$$

(d) About the Proof. As is the case with most evolution equations of the above kind, the derivation hinges on finding an appropriate Itô formula and applying it carefully. The following Itô formula, which is actually slightly more general than we require, and is modelled on a similar result of Dawson (1978), will be proven in Section 3.

Lemma 1.3 (Itô formula). *Let X_t be a Brownian or stable superprocess. Fix $k \ge 1$, $\Psi \in C^2(\Re_+ \times \Re^k)$, and assume Ψ and its first and second order derivatives satisfy a polynomial growth condition at infinity. Let $\varphi_1, \ldots, \varphi_k$ belong to the space $W^{2,p} \cap W^{2,1}$ for some $p \ge 2$. Let $\hat\varphi = (\varphi_1, \ldots, \varphi_k)$, and write $\langle \hat\varphi,\, X_t\rangle$ for the vector $(\langle \varphi_1, X_t\rangle, \ldots, \langle \varphi_k, X_t\rangle)$. Then, for all $t > 0$,*

$$\begin{aligned}\Psi(t, \langle \hat\varphi, X_t\rangle) &= \Psi(0, \langle \hat\varphi, X_0\rangle) + \int_0^t \Psi_t(s,\, \langle \hat\varphi, X_s\rangle)\, ds \\ &\quad + \int_0^t \sum_{i=1}^k \Psi_{x_i}(s,\, \langle \hat\varphi, X_s\rangle) \cdot \langle \Delta_\alpha \varphi_i,\, X_s\rangle\, ds \\ &\quad + \frac{1}{2} \int_0^t \sum_{i=1}^k \sum_{j=1}^k \Psi_{x_i x_j}(s,\, \langle \hat\varphi, X_s\rangle) \cdot \langle \varphi_i \varphi_j,\, X_s\rangle\, ds \\ &\quad + \int_0^t \int_{\Re^d} \sum_{i=1}^k \Psi_{x_i}(s,\, \langle \hat\varphi, X_s\rangle) \varphi_i(x)\; Z(ds, dx) \qquad (1.11)\end{aligned}$$

where

$$\begin{aligned}\Psi_t &= \partial\Psi(t,x)/\partial t, \\ \Psi_{x_i} &= \partial\Psi(t,x)/\partial x_i, \\ \Psi_{x_i x_j} &= \partial^2\Psi(t,x)/\partial x_i \partial x_j.\end{aligned}$$

As is usual, the Itô formula also holds if Ψ is a non-anticipative functional of the process X_t. We shall be interested in the particular functional

$$\Psi(t,x) = x\int_0^t \langle \psi, X_s\rangle\, ds, \tag{1.12}$$

where $\psi \in W^{2,p} \cap W^{2,1}$ and $x \in \Re^d$. Note that

$$\begin{aligned} \Psi(0,x) &\equiv 0,\\ \Psi_t(t,x) &= x\langle \psi, X_t\rangle,\\ \Psi_x(t,x) &= \int_0^t \langle \psi, X_s\rangle\, ds,\\ \Psi_{xx}(t,x) &\equiv 0. \end{aligned}$$

Apply (1.11) with this choice of Ψ to obtain

Lemma 1.4 *Let $\varphi, \psi \in W^{2,p} \cap W^{2,1}$ for some $p \geq 2$. Then for every finite $T > 0$,*

$$\begin{aligned} &\int_0^T \langle \psi(x)\varphi(y),\ X_t(dx)\, X_T(dy)\rangle\, dt \\ &= \int_0^T dt \int_0^t ds\, \langle \psi(x)\Delta_\alpha\varphi(y),\ X_s(dx)\, X_t(dy)\rangle \\ &\quad + \int_0^T \langle \psi(x)\varphi(y),\ X_t(dx)\, X_t(dy)\rangle\, dt \\ &\quad + \int_0^T \int_{\Re^d} \int_0^t ds\, \langle \psi(x)\varphi(y),\ X_s(dx)\rangle\ Z(dt,\, dy). \end{aligned} \tag{1.13}$$

This lemma is of importance in so far as it leads us to the following Lemma 1.5, which almost implies the central Theorem 1.2. Lemma 1.5, itself, follows in a reasonably straightforward fashion from Lemma 1.4 by first extending the latter to functions of the form $\Phi(x,y) = \sum_{i=1}^N \varphi_i(x)\psi_i(y)$ and then to general $\Phi(x,y) \in C^2(\Re^d \times \Re^d)$ of compact support.

Lemma 1.5 *Let $\Phi(x,y) \in C^2(\Re^d \times \Re^d)$ have compact support. Then (1.13) continues to hold if $\psi(x)\varphi(y)$ is replaced, throughout, by $\Phi(x,y)$, and $\psi(x) \times\Delta_\alpha\varphi(y)$ by $\Delta_\alpha^{(y)}\Phi(x,y)$, where we use $\Delta_\alpha^{(y)}$ to denote Δ_α operating on the second variable of $\Phi(x,y)$. That is, after reordering*

$$\begin{aligned} &\int_0^T dt \int_0^t ds\, \langle \Delta_\alpha^{(y)}\Phi(x,y),\ X_s(dx)\, X_t(dy)\rangle \\ &\quad + \int_0^T \langle \Phi(x,y),\ X_t(dx)\, X_t(dy)\rangle\, dt \\ &\qquad = \int_0^T \langle \Phi(x,y),\ X_t(dx)\, X_T(dy)\rangle\, dt \\ &\qquad\quad - \int_0^T \int_{\Re^d} \int_0^t ds\, \langle \Phi(x,y),\ X_s(dx)\rangle\, Z(dt,\, dy)\ . \end{aligned} \tag{1.14}$$

To derive the evolution equation (1.10) for the ILT, $\gamma(T,\varphi)$, "all" that has to be done is to replace the function $\Phi(x,y)$ of (1.14) by $G^{\lambda}_{\alpha}(x-y)\varphi(x)$, and to note the relationship (1.7) between G^{λ}_{α}, Δ_{α} and the delta function δ as well as the definition of the renormalized ILT as the limit of the $\gamma_{\epsilon}(T,\varphi)$ of (1.9). Unfortunately, however, this simple recipe requires justification at a large number of stages. Hence the long proof of Section 3.

Acknowledgements: Our path, through the calculations of the following two sections, was made substantially easier by being able to obtain guidance from the Master of the Moment, Jay Rosen. Steve Krone pointed out some minor errors in an earlier version of the paper.

2 On Evaluating Moments

An important component in virtually all the proofs of the following section will be the evaluation of moments of the kind $E\prod_{i=1}^{k}\langle\varphi_i,\, X_{t_i}\rangle$, when X_t is a Brownian or stable super process. An algorithm for calculating these, based exclusively on the transition densities $p_t(x,y)$ and $p_t^{\alpha}(x,y)$ of Section 1(c), has been given by Dynkin (1988)[2].

$$\begin{aligned} E&\left(\prod_{i=1}^{k}\langle\varphi_i,\, X_{t_i}\rangle\right) \\ &= \sum_{D_k}\int\prod_{v\in V_-} dy_v \prod_{\ell\in L} p^{\alpha}_{s_{f(\ell)}-s_{i(\ell)}}(y_{f(\ell)}-y_{i(\ell)}) \\ &\quad\times\prod_{v\in V_o} ds_v dy_v \prod_{i=1}^{k}\varphi_i(z_i)dz_i \end{aligned} \tag{2.1}$$

where $p_t^{\alpha}(x)\equiv 0$ if $t<0$, and D_k is the set of directed binary graphs with k exits marked $1,2,\ldots,k$. The convention that $p_t^{\alpha}(x)\equiv 0$ if $t<0$ is important, and will be used throughout the remainder of this paper without further comment. Given such a graph, L is the set of directed links, and if the link $\ell\in L$ goes from vertex v to vertex w, we write $v=i(\ell)$, $w=f(\ell)$. To each vertex v we associate two variables.

$$(s_v,y_v)\in\Re_+\times\Re^d\ ,$$

[2]Dynkin's formulae were acually proven for finite initial measures, and any positive, measurable φ. That the formulae also hold for a Lebesgue initial measure, and $\varphi\in\mathcal{L}^p$, $p>1$, is a consequence of Dynkin's proof and Theorem 2.3 of Adler and Lewin (1991).

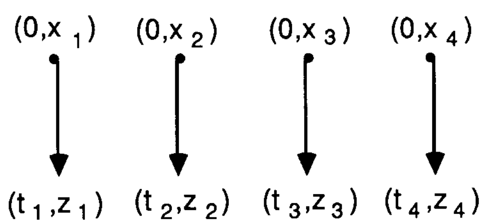
which we refer to as the time and space coordinates of v. V_- denotes the set of entrances for our graph, and if $v \in V_-$ we set $s_v = 0$ and $y_v = x_v$. If v is the exit labelled by j, $i \le j \le n$, we set

$$(s_v, y_v) \equiv (t_j, z_j).$$

Finally, V_o denotes the set of internal vertices, i.e. those vertices that are neither entrances nor exits.

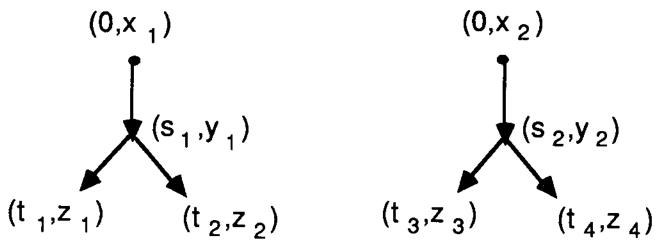
We shall, in what follows, be interested only in third and fourth order moments. In the latter case, the set D_4 of (2.1) consists of the following six basic graphs, and their various combinatorial rearrangements. The contribution of graph – i.e. the integral appearing in (2.1) – is written after the graph. Since we shall be concerned only with finiteness of moments, we shall not bother with the combinatorial factors associated with each graph.

Graph 1



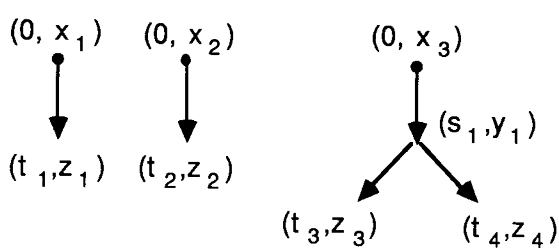
$$\prod_{i=1}^{4} \left\{ \int p_{t_i}^{\alpha}(x_i, z_i) \varphi_i(z_i)\, dx_i dz_i \right\}.$$

Graph 2



$$\int dx_1 dx_2 \int dy_1 dy_2 \int ds_1 ds_2\, p^{\alpha}_{s_1}(x_1, y_1) p^{\alpha}_{s_2}(x_2, y_2)$$
$$\times \int p^{\alpha}_{t_1 - s_1}(y_1, z_1) p^{\alpha}_{t_2 - s_1}(y_1, z_2) p^{\alpha}_{t_3 - s_2}(y_2, z_3) p^{\alpha}_{t_4 - s_2}(y_2, z_4) \prod_{i=1}^{4} \varphi(z_i) dz_i$$

Graph 3



$$\int dx_1 dx_2 dx_3 \int ds_1 \int dy_1\, p_{s_1}(x_3, y_1)$$
$$\times \int p^{\alpha}_{t_1}(x_1, z_1) p^{\alpha}_{t_2}(x_2, z_2) p^{\alpha}_{t_3 - s_1}(y_1, z_3) p^{\alpha}_{t_4 - s_1}(y_1, z_4) \prod_{i=1}^{4} \varphi(z_i) dz_i$$