CHAPTER I

1. Graphs and trees. A *tree* is a connected graph without circuits. This definition requires a word of explanation of the terms *graph*, *connected*, and *circuit*. A graph is a pair $(\mathfrak{X},\mathfrak{E})$ consisting of a set of *vertices* $\mathfrak{X}$ and a family $\mathfrak{E}$ of two-element subsets of $\mathfrak{X}$, called *edges*. When two vertices x, y belong to the same edge (i.e., $\{x,y\}\in\mathfrak{E}$) they are said to be *adjacent*; we also say that x and y are *nearest neighbors*.

A *path* in the graph $(\mathfrak{X},\mathfrak{E})$ is a finite sequence $x_0,\dots,x_n$, such that $\{x_i,x_{i+1}\}\in\mathfrak{E}$. A graph is called *connected* if, given two vertices $x,y\in\mathfrak{X}$, there exists a path $x_0,\dots,x_n$, with $x_0=x$ and $x_n=y$.

A *chain* is a path $x_0,\dots,x_n$, such that $x_i\neq x_{i+2}$, for $i=0,\dots,n-2$. A chain $x_0,\dots,x_n$, with $x_n=x_0$ is called a *circuit*. In particular if $(\mathfrak{X},\mathfrak{E})$ is a tree and $x,y\in\mathfrak{X}$, there exists a unique chain $x_0,\dots,x_n$, joining x to y. We denote this chain by $[x,y]$.

We are interested in *locally finite* trees. These are trees such that every vertex belongs to a finite number of edges. The number of edges to which a vertex x of a locally finite tree belongs is called the *degree* of x. If the degree is independent of the choice of x, then the tree is called *homogeneous*. In these notes we will treat mainly *locally finite homogeneous trees*.

The common degree of all vertices of a homogeneous tree is called the *degree of the tree* and is generally denoted by q+1. The reason for this notation is that, as will be shown in the Appendix, the number q may be identified, in many cases, with the order of a certain finite field. Furthermore many of the formulae appearing in the sequel, and especially in the explicit computation of spherical functions (Chapter **II**, below)

Cambridge University Press
978-0-521-42444-8 - Harmonic Analysis and Representation Theory for Groups Acting on Homogeneous Trees
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involve powers of q, rather than q+1.

There are also nonhomogeneous trees which are of interest. An important example is that of a *semihomogeneous tree*. Suppose that l and q are positive integers. A tree such that every vertex has degree l+1 or q+1, and such that two adjacent vertices have different degrees, is called *semihomogeneous* of degree (l,q).

The set of vertices of a homogeneous or semihomogeneous tree is always infinite. A tree may be represented graphically as shown in Figs 1 and 2.

q+1 = 3 q+1 = 4

Fig.1 Homogeneous trees

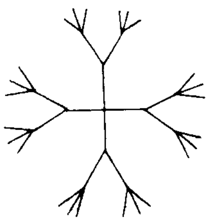

Fig.2. Semihomogeneous tree: q = 2, l = 3

The set of vertices of a tree is naturally a metric space.

The *distance* $\mathbf{d}(x,y)$ between any two distinct vertices x and y is defined as the number of edges in the chain $[x,y]$ joining x and y, in other words the *length* of $[x,y]$.

The metric space structure of the set of vertices $\mathfrak{X}$ suffices to define the tree uniquely because two vertices belong to the same edge if and only if their distance is 1. We will often think of a tree as a set of vertices with a metric which makes it into a tree.

An *infinite chain* is an infinite sequence $x_0,x_1,x_2,\dots,$ of vertices such that, for every i, $x_i \neq x_{i+2}$ and $\{x_i,x_{i+1}\}$ is an edge.

We define an equivalence relation on the set of infinite chains, by declaring two chains $x_0,x_1,x_2,\dots$ and $y_0,y_1,y_2,\dots$ equivalent if (as sets of vertices) they have an infinite intersection. This means that there is an integer $n\in\mathbb{Z}$ such that $x_k=y_{k+n}$ for every k sufficiently large. The *boundary* Ω of a tree is the set of equivalence classes of infinite chains. Observe that an infinite chain identifies uniquely a point of the boundary, which may be thought of as a point at infinity. Sometimes the points of the boundary are called *ends* of the tree.

An alternative way to define the boundary is by fixing a vertex x_0 and considering all infinite chains which start at x_0. A boundary point is associated with a unique infinite chain starting at x_0.

A *doubly infinite chain* is a sequence of vertices indexed by the integers, $\dots x_{-2},x_{-1},x_0,x_1,x_2,\dots,$ with the properties that $x_i \neq x_{i+2}$, and $\{x_i,x_{i+1}\}$ is an edge for every integer $i\in\mathbb{Z}$. A doubly infinite chain is also called an *infinite geodesic*. It identifies two boundary points. Conversely, given two distinct boundary points $\omega_1,\omega_2\in\Omega$, there is a unique geodesic joining them. We denote this geodesic by (ω_1,ω_2). We also use the notation $[x,\omega)$ for an infinite chain starting at x in the direction of ω (that is belonging to the equivalence class ω).

Since we also want to consider the direction of chains, infinite chains and geodesics, $(\omega, x]$ will be considered different from $[x, \omega)$ even though they have the same vertices, and similarly $[x,y]$ is formally different from $[y,x]$ and (ω_1, ω_2) is different from (ω_2, ω_1).

All these concepts are more or less geometrically evident and may be illustrated with a picture

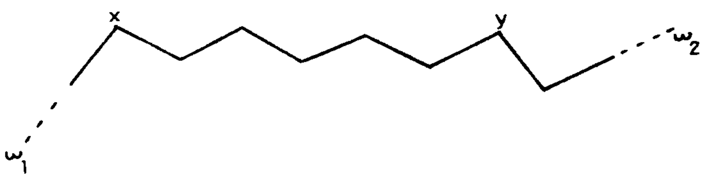

Fig.3 The boundary points ω_1 and ω_2 identify a geodesic (ω_1, ω_2)

The space $\mathfrak{X}\cup\Omega$ can be given a topology in which $\mathfrak{X}\cup\Omega$ is compact, the points of $\mathfrak{X}$ are open and $\mathfrak{X}$ is dense in $\mathfrak{X}\cup\Omega$. To define this topology it suffices to define a basis of neighborhoods for each boundary point (because each vertex is open). Let $\omega\in\Omega$, and let x be a vertex. Let $\gamma=[x,\omega)$ be the infinite chain from x to ω. For each $y\in[x,\omega)$ the neighborhood $\mathfrak{C}(x,y)$ of ω is defined to consist of all vertices and all end points of the infinite chains which include y but no other vertex of $[x,y]$ (Fig.4).

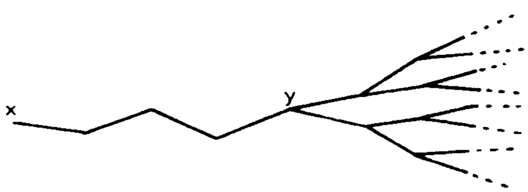

Fig.4

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It is not difficult to show that $\mathfrak{X}\cup\Omega$ is compact and $\mathfrak{X}$ is dense in Ω. The relative topology on Ω (under which Ω is compact) is best described by the open sets $\Omega(x,y)$ consisting of all boundary points associated to infinite chains which start at x and pass through y (in this order). In this way, for each vertex x, $\Omega(x,x)=\Omega$ and, for every positive integer n, $\Omega=\bigcup\{\Omega(x,y):\ \mathbf{d}(x,y)=n\}$. Thus the family $\{\Omega(x,y):\ \mathbf{d}(x,y)=n\}$ is a partition of Ω into $(q+1)q^{n-1}$ open and compact sets, where q+1 is the degree of the tree. Using these partitions we can define a measure ν_x on the algebra of sets generated by the sets $\Omega(x,y)$, by letting $\nu_x(\Omega(x,y))=1/(q+1)q^{n-1}$, if $\mathbf{d}(x,y)=n$. The positive measure ν_x may be extended to a Borel probability measure on Ω.

2. The free group as a tree. We preview in this section an example to which we will return in Section 7. Let $\mathbb{F}_2$ be the free group with two generators a and b. An element of $\mathbb{F}_2$ is a reduced word in the letters a,a^{-1},b,b^{-1}. We denote by e the empty word, which is the identity of $\mathbb{F}_2$. There is a natural correspondence between $\mathbb{F}_2$ and the vertices of a tree of degree 4, which is obtained by defining two words x and y to be in the same edge if $y^{-1}x$ is one of the generators or their inverses. This means that x and y can be obtained one from the other by right multiplication with an element of $\{a,a^{-1},b,b^{-1}\}$. The tree which is so defined is described by Fig.5

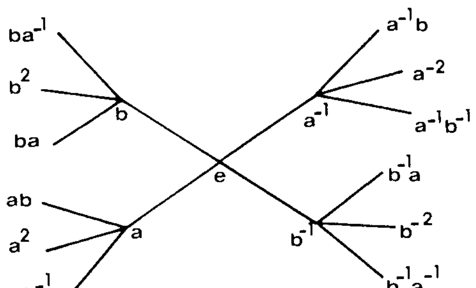

Fig.5

Other groups which may be similarly associated with homogeneous trees are described in Section 6, as certain discrete subgroups of the group of automorphisms of a tree.

It is interesting that the boundary of the tree of Fig.5 can be identified with the set of *infinite reduced words* in the letters $\{a,b,a^{-1},b^{-1}\}$. That is the words $\omega=x_1x_2x_3\ldots$, with $x_i\neq x_{i-1}^{-1}$ and $x_i\in\{a,b,a^{-1},b^{-1}\}$. We may also observe that left multiplication by elements of $\mathbb{F}_2$ on itself gives rise to an isometry of the tree. Left multiplication by a finite word is also defined on the infinite words and gives rise to a homeomorphism of Ω (see Section 7, below).

3. Automorphisms of a tree. An automorphism of a tree is a bijective map of the set of vertices onto itself which preserves the edges. An automorphism is also an isometry of the metric space $\mathfrak{X}$ endowed with the natural metric. Conversely, every isometry of $\mathfrak{X}$ is also an automorphism. We shall give presently a description of the automorphisms of a tree. We first need a preliminary result.

(3.1) *LEMMA. Let* g *be an automorphism of the tree and* x *a vertex. Let* $x=x_0,x_1,\ldots,x_n=g(x)$ *be the chain joining* x *to* $g(x)$, *and suppose that* $n>0$. *If* $g(x_1)\neq x_{n-1}$, *then there exists a doubly infinite chain,*

$$\{\ldots x_{-n},x_{-n+1},\ldots,x_{-1},x_0,x_1,\ldots,x_n=g(x_0),\ x_{n+1},\ldots\},$$

such that $g(x_j)=x_{j+n}$, *for every* $j\in\mathbb{Z}$.

PROOF. If $g(x_1)\neq x_{n-1}$ we can extend the chain $[x_0,g(x_0)]$ by defining $x_{n+1}=g(x_1)$. Since g is bijective, $g(x_0)=x_n$ implies that $g(x_2)\neq x_n$. Therefore we can define $g(x_2)=x_{n+2}$. By induction we define $g(x_k)=x_{j+k}$ and similarly $x_{j-k}=g^{-1}(x_{j-k})$. We obtain in this way a doubly infinite chain on which g acts according to the formula $g(x_j)=x_{n+j}$ for every $j\in\mathbb{Z}$. ∎

(3.2) *THEOREM. Let* g *be an automorphism of a tree; then one and only one of the following occurs.*

(1) g *stabilizes a vertex.*

(2) g *stabilizes an edge exchanging the vertices of the same edge.*

(3) *There exist a doubly infinite chain* $\gamma=\{x_n\}$ *and an integer* j *such that* $g(x_n)=x_{n+j}$ *for every* $n\in\mathbb{Z}$.

PROOF. Let g be any automorphism and let $j=\min\{\mathbf{d}(x,g(x)):\ x\in\mathfrak{X}\}$. Let $x\in\mathfrak{X}$ be such that $\mathbf{d}(x,g(x))=j$. If $j=0$, then $g(x)=x$ and g satisfies condition (1). If $j=1$, then $\{x,g(x)\}$ is an edge. In this case g satisfies (2) if $g^2(x)=x$, or (3) if $g^2(x)\neq x$ (this follows from (3.1)). Finally if $j>1$ and $[x,g(x)]=\{x_0=x, x_1,\ldots,x_j=g(x)\}$, then $g(x_1)\neq x_{j-1}$, because $\mathbf{d}(x_1,x_{j-1})=(j-2)<j$. Therefore g satisfies condition (3). The fact that only one of the three conditions holds follows readily. ∎

On the basis of (3.2) it is natural to give the following definition.

(3.3) *DEFINITION. An automorphism of a tree is called a rotation if it stabilizes a vertex; an inversion if it satisfies condition* (2) *of* (3.2)*; and a translation of step* j *along* γ *if it satisfies condition* (3) *of* (3.2). *In this last case* $j=\min\{\mathbf{d}(x,g(x)):\ x\in\mathfrak{X}\}$ *and* $\gamma=\{x\in\mathfrak{X}:\mathbf{d}(x,g(x))=j\}$.

If the tree is homogeneous there are always rotations, inversions and translations of any step on any geodesic. For a locally finite nonhomogeneous tree the situation may be quite different. First of all the existence of a translation implies the existence of an infinite geodesic. The only possible automorphisms of a finite tree are rotations and inversions. Furthermore, for any automorphism g, the degree of x and g(x) is the same. This implies, for instance, that in a semihomogeneous tree $g(a)\neq b$ if $\{a,b\}$ is an edge. In other words every automorphism of a semihomogeneous tree is either a rotation or a translation of even step.

We remark that, if g and g' are automorphisms of a tree, then g' and $gg'g^{-1}$ are both rotations, both inversions or both translations. It is also obvious that an m-power of a step-j translation on γ is a step-$|mj|$ translation on γ, for every $m\in\mathbb{Z}$.

We conclude with a technical result which will be useful later.

(3.4) *PROPOSITION.* (1) *The composition of two inversions on distinct edges is a translation of even step.*
(2) *The composition of an inversion about an edge and a rotation which does not fix both vertices of the edge is a translation of odd step on a geodesic containing that edge.*
(3) *Every automorphism of a homogeneous tree of degree* q+1>2 *is a product of translations.*
PROOF. Let {a,b} and {c,d} be distinct edges, that is, having at most one vertex in common (Fig.6).

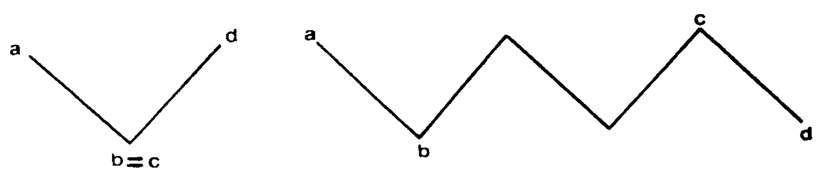

Fig.6

Let h be an inversion on {a,b} and g be an inversion on {c,d}. If b=c, then gh(a)=g(b)=d and gh(b)=g(a)≠b (because a≠d=g^{-1}(b)). This means that gh(b) is a vertex of distance 1 from d which is not b. By (3.1) gh is a step-2 translation on a geodesic γ containing {a,b,d}. Suppose that the two edges {a,b} and {c,d} have no vertex in common. By naming the four vertices appropriately we may suppose that $\mathbf{d}(b,c)=n$ is the distance of the two edges. This means that the chain $[a,d]$ contains $[b,c]$. Then $g(b)\notin[b,d]$ and $g(a)\notin[b,g(b)]$ (Fig.6). This implies by (3.1) that gh is a translation on a geodesic containing $[a,g(a)]$. Finally we observe that $\mathbf{d}(d,g(b))=\mathbf{d}(g(c),g(b))=n$ and therefore $\mathbf{d}(a,gh(a))=\mathbf{d}(a,g(b))=\mathbf{d}(a,b)+\mathbf{d}(b,c)+\mathbf{d}(c,d)+\mathbf{d}(d,g(b))=2(n+1)$, which means that gh is of even step. To prove (2) let g be an inversion on the edge $\{a,b\}$ and k a rotation. If $k(a)=a$ and $k(b)\neq b$, then $kg(b)=k(a)=a$, while $kg(a)=k(b)\neq b$. By (3.1), applied to $[a,b]$, we conclude that kg is a translation of step 1. If $k(a)\neq a$, and $k(b)\neq b$, let x be the point of minimal distance from $\{a,b\}$ such that $k(x)=x$. Suppose that $\mathbf{d}(x,b)=\mathbf{d}(x,a)+1$. Then $\mathbf{d}(k(a),b)=\mathbf{d}(k(a),a)+1=2\mathbf{d}(x,a)+1$. Now $k(g(b))=k(a)$ and $kg(a)=k(b)\notin[b,k(a)]$. Therefore by (3.1) applied to $[b,k(a)]$ we conclude that kg is a translation of step $2\mathbf{d}(x,a)+1$ (Fig.7).

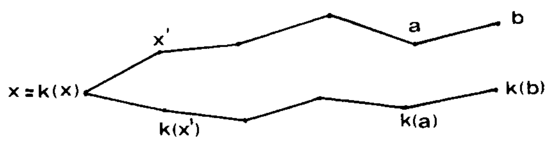

Fig.7

This proves (2). Finally assume that the tree is homogeneous with degree $q+1>2$. Let g be an inversion on the edge $\{a,b\}$ and let γ be a doubly infinite geodesic containing a but not b. (such a γ exists because $q+1>2$). Let τ be a step-1 translation on γ. Then $\tau(b)\neq a$ and $\tau g(a)=\tau(b)$ (Fig.8).

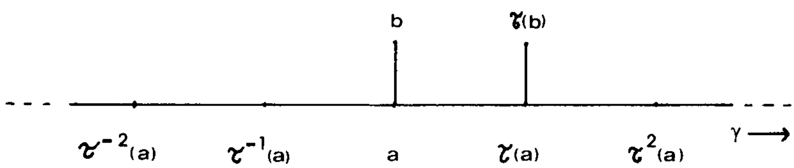

Fig.8

We apply now (3.1) to $[b,\tau(a)]$ to conclude that τg is a step-1 translation. This shows that $g=\tau^{-1}\tau g$ is the composition of two translations. Let now k be a nontrivial rotation. Since k is nontrivial and fixes a point x_0, it maps a finite chain $\mathscr{C}$ starting at x_0 into a different finite chain $k(\mathscr{C})$, also starting at x_0. Let a be the first point of $\mathscr{C}$ such that $a\neq k(a)=b$ and let x be the last point of $\mathscr{C}$ such that $k(x)=x$; then $\mathbf{d}(x,a)=\mathbf{d}(x,b)=1$. Let g be an inversion on $\{x,a\}$. Then $gk^{-1}(b)=x$ and $gk^{-1}(x)=a$, that is $\tau=gk^{-1}$ is a step-1 translation. Therefore $k=\tau^{-1}g$ is the product of a translation and an inversion. Since every inversion is the product of translations, (3) follows. ∎

4. The group of automorphisms Aut($\mathfrak{X}$). We assume from now on that $(\mathfrak{X},\mathfrak{E})$ is a homogeneous tree of degree q+1, and we let Aut($\mathfrak{X}$) denote the group of automorphisms of $(\mathfrak{X},\mathfrak{E})$, which is the same as the group of isometries of $\mathfrak{X}$ as a metric space.

It is not difficult to define on Aut($\mathfrak{X}$) a *locally compact topology* under which the group operations are continuous. To define a basis of neighborhoods of $g\in\mathrm{Aut}(\mathfrak{X})$, let F be a finite subset of $\mathfrak{X}$, and let $U_F(g)=\{h\in\mathrm{Aut}(\mathfrak{X})\colon g(x)=h(x), \text{ for all } x\in F\}$. It is clear that, under the topology generated by the sets $U_F(g)$, the group operations of Aut($\mathfrak{X}$) are continuous. It is also not difficult to show that the topology is *locally compact*. Indeed, for $x\in\mathfrak{X}$, let $K_x=\{g\in\mathrm{Aut}(\mathfrak{X})\colon g(x)=x\}$. By definition K_x is open. But K_x is also compact as will be presently shown. Every $g\in K_x$ acts as a permutation on the set $\mathfrak{B}_n=\{w\in\mathfrak{X}\colon \mathbf{d}(x,w)=n\}$, the set of vertices of distance n from x. This set $\mathfrak{B}_n$ has $r_n=(q+1)q^{n-1}$ elements. Therefore K_x may be thought of as a subgroup of the infinite product of the permutation groups $S(r_n)$. We will show that in this product K_x is closed. An element $g\in\prod S(r_n)$ is in the complement of K_x if, for some n, and some $w_1,w_2\in\mathfrak{B}_n$, $g(w_1)=w_2$, while some element in the chain between x and w_1 is not mapped by g into the element