Homologically finite subcategories

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Let Λ be an artin algebra and modΛ the category of finitely generated Λ-modules. Unless stated to the contrary, by a subcategory $\mathcal{C}$ of a category we mean a full subcategory of an additive category, for example modΛ, closed under isomorphisms and direct summands. About ten years ago, in connection with proving the existence of preprojective and preinjective partitions for modΛ as well as the existence of almost split sequences in certain subcategories of modΛ, Auslander and Smalø introduced the notions of contravariantly, covariantly and functorially finite subcategories of modΛ and developed some of their basic properties [5] [6]. We refer to the study of these subcategories as the theory of homologically finite subcategories. At the time Auslander and Smalø pointed out that there is an intimate connection between the tilting theory of Happel and Ringel based on tilting modules of projective dimension at most one and the theory of homologically finite subcategories [6]. Utilizing recent developments in the theory, we have now given precise connections between the tilting theory developed by Miayshita and Happel based on tilting modules of arbitrary finite projective dimension and the theory of homologically finite subcategories in [4]. Inspired by this new point of view on tilting theory, Ringel has shown that associated with a quasihereditary algebra Λ and a particular type of ordering of its simple modules is a naturally defined tilting module whose endomorphism ring is again quasihereditary. It follows from this that quasihereditary algebras occur in pairs [13].

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In addition to tilting theory, questions in the theory of homologically finite subcategories have come up in the work of Burt and Butler in showing that some categories of representations of bocses have almost split sequences [8]. This problem was investigated by Bautista-Kleiner in [7] using other methods.

More generally, suppose that Λ is a noetherian R-algebra, i.e. R is a commutative noetherian ring and Λ is an R-algebra which is a finitely generated R-module. There are interesting anologues for noetherian R-algebras of much of the theory of homologically finite subcategories of modΛ when Λ is an artin algebra. In particular, the Auslander–Buchweitz theory of Cohen-Macaulay approximations is proving to be of special interest in the study of complete local Cohen-Macaulay rings [2]. It is also worth noting that the theory of Cohen-Macaulay approximations in general has helped clarify some points in the theory of homologically finite subcategories for artin algebras. However, for the most part we will restrict our attention in this paper to the artin algebra situation.

In section 1 we recall the definitions of contravariantly, covariantly and functorially finite subcategories, and investigate the connections with adjoint functors. In section 2 we discuss the origins of the concepts, in connection with preprojective partitions and almost split sequences in subcategories. In section 3 we study ways of constructing covariantly finite subcategories from contravariantly finite ones, and conversely. In section 4 we give the connections with tilting theory.

In general proofs will only be given when they have not already appeared elsewhere. Usually also dual definitions and statements will be left to the reader.

1 Homologically finite subcategories and adjoint functors

Even though most of this paper is concerned with full subcategories of categories of finitely generated modules over artin algebras, we begin by considering homologically finite subcategories of arbitrary categories. In this way we hope to emphasize the generality of the basic notions of the theory of homologically finite subcategories.

This section is devoted to giving the definitions of contravariantly, covariantly and functorially finite subcategories of arbitrary categories and their connections with adjoint functors. We begin by giving some basic definitions and notation [5].

Let $\mathcal{C}$ be an arbitrary category, not necessarily a category of modules. For each pair of objects C_1 and C_2 in $\mathcal{C}$ we denote the set of morphisms from C_1 to C_2 by $\mathcal{C}(C_1, C_2)$. For each C in $\mathcal{C}$ we denote by $\mathcal{C}(\ ,C)$ the contravariant functor from $\mathcal{C}$ to Sets given by $X \mapsto \mathcal{C}(X,C)$ for all objects X in $\mathcal{C}$ and we denote by $\mathcal{C}(C,\ )$ the covariant functor from $\mathcal{C}$ to Sets given by $X \mapsto \mathcal{C}(C,X)$ for all objects X in $\mathcal{C}$. A contravariant (covariant) functor $F : \mathcal{C} \to \text{Sets}$ is said to be representable if $F \cong \mathcal{C}(\ ,C)$ ($F \cong \mathcal{C}(C,\ )$) for some object C in $\mathcal{C}$. A morphism $\alpha : F \to G$ of functors from $\mathcal{C}$ to Sets is said to be surjective if $\alpha_X : F(X) \to G(X)$ is surjective for all objects X in $\mathcal{C}$. A contravariant (covariant) functor F from $\mathcal{C}$ to Sets is said to be finitely generated if there is a surjection $\mathcal{C}(\ ,C) \to F$ ($\mathcal{C}(C,\ ) \to F$) for some C in $\mathcal{C}$.

Suppose $\mathcal{X}$ is a subcategory of $\mathcal{C}$. By a right $\mathcal{X}$-approximation of an object C in $\mathcal{C}$ we mean a morphism $f_C : X_C \to C$ with X_C an object in $\mathcal{X}$ such that the induced morphism $\mathcal{C}(X, X_C) \to \mathcal{C}(X,C)$ is surjective for all objects X in $\mathcal{X}$. It is not difficult to see that an object C in $\mathcal{C}$ has a right $\mathcal{X}$-approximation if and only if $\mathcal{C}(\ ,C)\,|_{\mathcal{X}}$, the restriction to the subcategory $\mathcal{X}$ of the functor $\mathcal{C}(\ ,C)$, is finitely generated, i.e. there is a surjection $\mathcal{X}(\ ,X_C) \to \mathcal{C}(\ ,C)\mid_{\mathcal{X}}$ for some object X_C in $\mathcal{X}$. We denote by $\operatorname{Rapp}\mathcal{X}$ the subcategory of $\mathcal{C}$ consisting of all the objects C in $\mathcal{C}$ which have right $\mathcal{X}$-approximations. Clearly, $\mathcal{X}$ is contained in $\operatorname{Rapp}\mathcal{X}$ and it is easily checked that $\operatorname{Rapp}\mathcal{X} = \operatorname{Rapp}(\operatorname{Rapp}\mathcal{X})$. Finally $\mathcal{X}$ is said to be contravariantly finite in $\mathcal{C}$ if $\operatorname{Rapp}\mathcal{X} = \mathcal{C}$, i.e. every object C in $\mathcal{C}$ has a right $\mathcal{X}$-approximation.

Dually, by a left $\mathcal{X}$-approximation of an object C in $\mathcal{C}$ we mean a morphism $g : C \to X^C$ with X^C in $\mathcal{X}$ such that the induced morphism $\mathcal{C}(X^C, X) \to \mathcal{C}(C, X)$ is surjective for all objects X in $\mathcal{C}$. It is not difficult to see that an object C in $\mathcal{C}$ has a left $\mathcal{X}$-approximation if and only if the retriction $\mathcal{C}(C,\ )\mid_{\mathcal{X}}$ is finitely generated. We denote by $\operatorname{Lapp}\mathcal{X}$ the subcategory of $\mathcal{C}$ consisting of all the objects C in $\mathcal{C}$ which have left $\mathcal{X}$-approximations. Clearly $\mathcal{X}$ is contained in $\operatorname{Lapp}\mathcal{X}$ and it is easily checked that $\operatorname{Lapp}\mathcal{X} = \operatorname{Lapp}(\operatorname{Lapp}\mathcal{X})$. We say that $\mathcal{X}$ is covariantly finite in $\mathcal{C}$ if $\operatorname{Lapp}\mathcal{X} = \mathcal{C}$. Finally we say that $\mathcal{X}$ is functorially finite in $\mathcal{C}$ if it is both contravariantly and covariantly finite in $\mathcal{C}$.

The rest of this section is devoted to explaining various connections between adjoint functors and contravariantly and covariantly finite subcategories which give nontrivial interesting examples of homologically finite subcategories.

Suppose $G : \mathcal{C} \to \mathcal{X}$ is a right adjoint for the inclusion functor $\operatorname{inc} : \mathcal{X} \to \mathcal{C}$. Then there are isomorphisms $\mathcal{C}(X, C) \to \mathcal{X}(X, G(C))$ for all X in $\mathcal{X}$ and C in $\mathcal{C}$ which are functorial in X and C. Therefore for each C in $\mathcal{C}$ we have that the restriction $\mathcal{C}(\ ,C)\mid_{\mathcal{X}}$ is representable in X since it is isomorphic to the representable functor $\mathcal{X}(\ ,G(C))$. It is also not difficult to see that $\operatorname{inc} : \mathcal{X} \to \mathcal{C}$ has a right adjoint if $\mathcal{C}(\ ,C)\mid_{\mathcal{X}}$ is representable in $\mathcal{X}$ for all C in $\mathcal{C}$. Since representable functors are finitely generated, it follows that if $\operatorname{inc} : \mathcal{X} \to \mathcal{C}$ has a right adjoint, then $\mathcal{X}$ is contravariantly finite in $\mathcal{C}$. Thus

$\mathcal{X}$ being contravariantly finite in $\mathcal{C}$ can be viewed as a generalization of the inclusion $\operatorname{inc} : \mathcal{X} \to \mathcal{C}$ having a right adjoint.

A similar argument shows that $\operatorname{inc} : \mathcal{X} \to \mathcal{C}$ having a left adjoint implies that $\mathcal{X}$ is covariantly finite in $\mathcal{C}$. Thus $\mathcal{X}$ being covariantly finite in $\mathcal{C}$ can be viewed as a generalization of $\operatorname{inc} : \mathcal{X} \to \mathcal{C}$ having a left adjoint.

Summarizing our discussion we have the following.

Proposition 1.1 *Let $\mathcal{X}$ be a subcategory of $\mathcal{C}$.*

a) *If $\operatorname{inc} : \mathcal{X} \to \mathcal{C}$ has a right adjoint, then $\mathcal{X}$ is contravariantly finite in $\mathcal{C}$.*

b) *If $\operatorname{inc} : \mathcal{X} \to \mathcal{C}$ has a left adjoint, then $\mathcal{X}$ is covariantly finite in $\mathcal{C}$.*

We now discuss some more connections between adjoint functors and contravariantly (covariantly) finite subcategories. Amongst other things, this discussion will show that there are contravariantly (covariantly) finite subcategories $\mathcal{X}$ of $\mathcal{C}$ such that $\operatorname{inc} : \mathcal{X} \to \mathcal{C}$ does not have a right (left) adjoint.

Suppose now that we are given categories $\mathcal{C}$ and $\mathcal{D}$ and a pair of adjoint functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. Then we have isomorphisms $\mathcal{D}(F(C), D) \xrightarrow{\sim} \mathcal{C}(C, G(D))$ for all C in $\mathcal{C}$ and D in $\mathcal{D}$ which are functorial in C and D. Associated with the functor F is the subcategory $\operatorname{Im} F$ of $\mathcal{D}$ consisting of all D in $\mathcal{D}$ isomorphic to $F(C)$ for some C in $\mathcal{C}$ and associated with the functor G is the subcategory $\operatorname{Im} G$ of $\mathcal{C}$ consisting of all C in $\mathcal{C}$ isomorphic to $G(D)$ for some D in $\mathcal{D}$. We now show that $\operatorname{Im} F$ is contravariantly finite in $\mathcal{D}$. The dual argument, which we do not give, shows that $\operatorname{Im} G$ is covariantly finite in $\mathcal{C}$.

Let D be in $\mathcal{D}$. Denote by $f : F(G(D)) \to D$ the map corresponding to the identity $G(D) \to G(D)$ under the adjointness isomorphism $\mathcal{D}(FG(D), D) \xrightarrow{\sim} \mathcal{C}(G(D), G(D))$. Suppose X is in $\operatorname{Im} F$ and we are given a morphism $g : X \to D$. We want to show that g factors through f. Since X is in $\operatorname{Im} F$, we have that $X \simeq F(C)$ for some C in $\mathcal{C}$. Let $h : C \to G(D)$ be the morphism corresponding to $g : F(C) \to D$ under the isomorphism $(F(C), D) \xrightarrow{\sim} (C, G(D))$. Then we obtain the map $F(h) : F(C) \to FG(D)$ which is well known to have the property $fF(h) = g$. Hence the morphism $f : FG(D) \to D$ induces a surjection $\mathcal{D}(X, FG(D)) \to \mathcal{D}(X, D)$ for all X in $\operatorname{Im} F$. Thus we have shown that $\operatorname{Im} F$ is contravariantly finite in $\mathcal{D}$.

Summarizing our discussion we have the following.

Proposition 1.2 *Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ an adjoint pair of functors with F a left adjoint and G a right adjoint. Then we have the following.*

a) $\operatorname{Im} F$ *is contravariantly finite in $\mathcal{D}$ and the usual morphism $FG \to I$ gives a right* $\operatorname{Im} F$*-approximation $FG(D) \to D$ for each D in $\mathcal{D}$.*

b) $\operatorname{Im} G$ *is covariantly finite in $\mathcal{C}$ and the usual morphism $I \to GF$ gives a left* $\operatorname{Im} G$*-approximation $C \to GF(C)$ for each C in $\mathcal{C}$.*

We now want to investigate when $\operatorname{inc} : \operatorname{Im} F \to \mathcal{D}$ and $\operatorname{inc} : \operatorname{Im} G \to \mathcal{C}$ have right and left adjoints respectively. Our results along these lines are based on the following.

Proposition 1.3 *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ an adjoint pair of functors with F a left adjoint and G a right adjoint. A functor $H : \mathcal{D} \to \operatorname{Im} F$ is a right adjoint of* $\operatorname{inc} : \operatorname{Im} F \to \mathcal{D}$ *if and only if there is a morphism* $\operatorname{inc} H \to I_{\mathcal{D}}$ *such that the induced morphism* $G \operatorname{inc} H \to G$ *is an isomorphism.*

Proof: Suppose $H : \mathcal{D} \to \operatorname{Im} F$ is a right adjoint of $\operatorname{inc} : \operatorname{Im} F \to \mathcal{D}$. Then there is a morphism $H \to I_{\mathcal{D}}$ with the property that the induced morphisms

$\mathcal{D}(U, H(D)) \to \mathcal{D}(U, D)$ are isomorphisms for all U in $\operatorname{Im} F$ and D in $\mathcal{D}$. Let D be in $\mathcal{D}$. Then the morphism $H(D) \to D$ gives rise to the morphism $GH(D) \to G(D)$ which in turn gives rise to the commutative diagram

$$\begin{array}{ccc} \mathcal{C}(C, GH(D)) & \to & \mathcal{C}(C, G(D)) \\ \downarrow\wr & & \downarrow\wr \\ \mathcal{D}(F(C), H(D)) & \xrightarrow{\sim} & \mathcal{D}(F(C), D) \end{array}$$

for all C in $\mathcal{C}$. Hence the morphism $GH(D) \to G(D)$ is an isomorphism since the induced morphisms $\mathcal{C}(C, GH(D)) \to \mathcal{C}(C, G(D))$ are isomorphisms for all C in $\mathcal{C}$. Therefore the morphism $H \to I_{\mathcal{D}}$ has the property that the induced morphism $G \operatorname{inc} H \to G$ is an isomorphism. □

As an immediate consequence of this result we have the following which we leave to the reader to prove.

Corollary 1.4 *If* $\operatorname{inc} : \operatorname{Im} F \to \mathcal{D}$ *has a right adjoint, then* $G(\operatorname{Im} F) = \operatorname{Im} G$.

Remark It would be interesting to know if the converse of Corollary 1.4 is true.

As another immediate consequence of Proposition 1.3 we have the following.

Corollary 1.5 *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be categories and* $F : \mathcal{C} \to \mathcal{D}$ *and* $G : \mathcal{D} \to \mathcal{C}$ *an adjoint pair of functors with* F *a left adjoint and* G *a right adjoint. Suppose* $G : \mathcal{D} \to \mathcal{C}$ *has the property that a morphism* $f : D_1 \to D_2$ *in* $\mathcal{D}$ *is an isomorphism whenever* $G(f) : G(D_1) \to G(D_2)$ *is an isomorphism. Then* $\operatorname{inc} : \operatorname{Im} F \to \mathcal{D}$ *has a right adjoint if and only if* $\operatorname{Im} F = \mathcal{D}$.

Proof: Suppose inc : $\operatorname{Im} F \to \mathcal{D}$ has a right adjoint. Then there is functor $H : \mathcal{D} \to \operatorname{Im} F$ giving a morphism $\operatorname{inc} H \to I_{\mathcal{D}}$ such that $G \operatorname{inc} H \to G$ is an isomorphism. But then $\operatorname{inc} H \to I_{\mathcal{D}}$ is an isomorphism, which shows that $\operatorname{Im} F = \mathcal{D}$. □

We now apply Corollary 1.5 in some special cases to determine if inc : $\mathcal{X} \to \mathcal{C}$ has a right adjoint when $\mathcal{X}$ is contravariantly finite in $\mathcal{C}$.

Consider the adjoint pair of functors F : Sets → Groups and G : Groups → Sets where $F(S)$ is the free group generated by the set S and G is the forgetful functor. Clearly G has the property that a morphism f in Groups is an isomorphism if $G(f)$ is an isomorphism. Therefore by Corollary 1.5, if inc : $\operatorname{Im} F \to$ Groups has a right adjoint, then $\operatorname{Im} F =$ Groups, which is impossible since $\operatorname{Im} F$ is the subcategory of Groups consisting of the free groups. Therefore the subcategory of Groups consisting of the free groups is contravariantly finite in Groups without the inclusion having a right adjoint.

The following is perhaps a more interesting example for ring and module theorists.

Proposition 1.6 *Let $f : \Lambda \to \Gamma$ be a map of rings with the property that Γ is a finitely presented left Λ-module. Let* $\operatorname{mod}\Lambda$ *and* $\operatorname{mod}\Gamma$ *be the categories of finitely presented Λ and Γ-modules respectively. Finally, let $F : \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$ be given by $F(M) = \Gamma \underset{\Lambda}{\otimes} M$ for all Λ-modules M. Then $\operatorname{Im} F$, the subcategory of* $\operatorname{mod}\Gamma$ *of induced modules, is contravariantly finite in* $\operatorname{mod}\Lambda$ *and* inc : $\operatorname{Im} F \to \operatorname{mod}\Gamma$ *has a right adjoint if and only if* $\operatorname{Im} F = \operatorname{mod}\Gamma$, *i.e. every finitely presented Γ-module is induced.*

Proof: Suppose M is in $\operatorname{mod}\Gamma$. Then res M, the restriction of M to Λ is in $\operatorname{mod}\Lambda$ since Γ is a finitely presented Λ-module. But then res : $\operatorname{mod}\Gamma \to \operatorname{mod}\Lambda$

is a right adjoint to $\Gamma \underset{\Lambda}{\otimes}$ with the property that $f : M_1 \to M_2$ in $\mathrm{mod}\Gamma$ is an isomorphism if $\mathrm{res}\, f : \mathrm{res}\, M_1 \to \mathrm{res}\, M_2$ is an isomorphism. Therefore we can apply Corollary 1.5 to get our desired result. □

It is obvious that if every Γ-module is induced from Λ, then the relative global dimension of the ring morphism $f : \Lambda \to \Gamma$ is zero. This already shows that the inclusion of the induced modules rarely has a right adjoint. However there are examples where the inclusion of the induced modules has no right adjoint even though the relative global dimension is zero. While there are examples where every Γ-module is induced, for instance if $f : \Lambda \to \Gamma$ is a ring epimorphism, there do not seem to be any good criteria for when this happens.

Let Λ be a twosided noetherian ring. We say that a Λ-module K is a d^{th} syzygy module ($d \geq 1$) if there is an exact sequence $0 \to K \to P_{d-1} \to \cdots \to P_0 \to C \to 0$ with the P_i projective Λ-modules. We end this section by showing that the category $\Omega^d(\mathrm{mod}\Lambda)$ of d^{th} syzygy modules is covariantly finite in $\mathrm{mod}\Lambda$. This was essentially first proved in [1]. But before doing this, it is convenient to introduce some notation.

Suppose Λ is a twosided noetherian ring and $d \geq 1$. Let $\underline{\mathrm{mod}}\Lambda$ denote the category of finitely generated Λ-modules modulo projectives. Then we denote by $\Omega^d : \underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda$ the d^{th} syzygy functor and by $Tr : \underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda^{op}$ the usual transpose. For A and B in $\underline{\mathrm{mod}}\Lambda$, we denote $\underline{\mathrm{mod}}(\Lambda)(A, B)$ by $\underline{\mathrm{Hom}}_\Lambda(A, B)$. We now prove the following.

Proposition 1.7 *For each $d \geq 1$, the functor $Tr\Omega^d Tr : \underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda$ is a left adjoint for $\Omega^d : \underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda$. Then the usual morphism $I_{\underline{\mathrm{mod}}\Lambda} \to \Omega^d(Tr\Omega^d Tr)$ has the property that $M \to \Omega^d(Tr\Omega^d Tr)(M)$ is a left $\mathrm{Im}\,\Omega^d$-approximation of M for all M in $\underline{\mathrm{mod}}\Lambda$. Therefore $\Omega^d(\underline{\mathrm{mod}}\Lambda)$ is covariantly finite in $\underline{\mathrm{mod}}\Lambda$.*

Proof: In view of Proposition 1.2, it suffices to prove that $Tr\Omega^d Tr$ is a left adjoint for Ω^d. Now we have isomorphisms $\underline{\mathrm{Hom}}(A, \Omega^d B) \xrightarrow{\sim} \mathrm{Tor}_1^\Lambda(TrA, \Omega^d B)) \xrightarrow{\sim} \mathrm{Tor}_1^\Lambda(\Omega^d TrA, B) \xrightarrow{\sim} \underline{\mathrm{Hom}}(Tr\Omega^d TrA, B)$ which are functorial in A and B [3], giving our desired result. □

As a fairly straightforward consequence of this result we have the following.

Corollary 1.8 $\Omega^d(\mathrm{mod}\Lambda)$ *is covariantly finite in* $\mathrm{mod}\Lambda$ *for each* $d \geq 0$.

Proof: When Λ is an artin algebra, it was shown in [5] that if $\mathcal{Y}$ is a covariantly finite subcategory of $\underline{\mathrm{mod}}(\Lambda)$ then the subcategory of $\mathrm{mod}\Lambda$ consisting of all M such that $M \cong Y$ in $\underline{\mathrm{mod}}(\Lambda)$ for some Y in $\mathcal{Y}$ is also covariantly finite. The proof works equally well for Λ a twosided noetherian ring using the fact that $\mathrm{Hom}_\Lambda(A, \Lambda)$ is a finitely generated Λ^{op}-module for all A in $\mathrm{mod}\Lambda$. □

We also cite the following result from [1] (see [10]).

Proposition 1.9 *Let* Λ *be an Auslander ring, that is, in the minimal injective resolution* $0 \to \Lambda \to I_0 \to I_1 \to \cdots \to I_i \to \cdots$ *of* Λ*, flat dim.* $I_j \leq j$ *for all* j*. Then for each* t*,* $\{C; \mathrm{pd}_\Lambda C \leq t\}$ *is covariantly finite in* $\mathrm{mod}\Lambda$.

2 Connections with preprojective partitions

Throughout this section all rings are artin algebras and all categories are subcategories of categories of finitely generated modules over an artin algebra. Our aim in this section is to review how the notion of homologically finite subcategories of $\mathrm{mod}\Lambda$ arose originally in [5] in connection with proving the