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Topics in Varieties of Group Representations

Samuel M. Vovsi
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To my father

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PREFACE

The study of varieties of algebraic structures, i.e. classes of algebraic structures definable by identical relations, was originated by G. Birkhoff [7] and B. H. Neumann [67] in the 1930's. A wide expansion of the ideas and methods of variety theory began in the 1950's, when the work of G. Higman, A. I. Mal'cev, B. H. Neumann, H. Neumann, A. Tarski, W. Specht was particularly influential. Since then the intensity of work in this area of algebra has remained very high, and the number of publications devoted to identities and varieties of algebraic structures is now counted in the thousands. Various aspects of the field have been systematically presented in numerous monographs and surveys — see for example [3, 12, 15, 16, 18, 64, 68, 80, 83, 84, 85, 86, 105].

As a result of this expansion, at the present moment one can speak of a number of independent but closely related algebraic theories: varieties of groups, varieties of associative algebras and polynomial identities, varieties of Lie algebras, varieties of semigroups, varieties of lattices and universal algebras, and others. We say "independent" because each of these fields has its own motivations and stimuli for development and its own natural problems. On the other hand, they are developing in close interconnection and are constantly influencing one another.

The present book is devoted to one of the newest branches of variety theory: varieties of group representations. This subject has existed for about twenty years; its foundations were laid in papers of B. I. Plotkin and his students in the late 1960's–early 1970's. There are many motivations for the study of varieties of group representations. First, from the standpoint of universal algebra all representations of groups over a given commutative ring form a variety of two-sorted algebras, with quite natural free objects, verbal subobjects and other standard attributes of variety theory. Second, a number of classical theorems and problems are, in fact, concerned with group representations satisfying certain identical relations and, as a result, can be naturally interpreted in the framework of the theory of varieties. Here one can mention a theorem of Kolchin [42] on triangulability of a unipotent matrix group, a theorem of Kaloujnine [38] on nilpotency of a stable group of automorphisms, and a series of problems and results concerning the augmentation ideal and dimension subgroups. Third, the theory of varieties of group representations has numerous connections with varieties of groups, varieties of associative and Lie algebras, group rings, etc., which can provide important applications in both directions. This was first demonstrated by Bryce [9] and Ol'shansky [71] who applied the technique of varieties of group representations to investigating varieties of abstract groups.

However, in contrast with "usual" algebras — groups, rings, etc. — representations of groups are *two-sorted* algebraic structures, i.e. they have two underlying sets. Therefore the theory of varieties of group representations exibits a number of characteristic features and essentially new problems. During the last decades the importance of many-sorted algebraic structures has been constantly increasing, largely because of numerous applications in the theory of automata, data banks, theoretical computer science, etc. Group representations are, probably, the most comprehensively studied objects of this type, and certain methods of the theory of varieties of group representations can be (and have been) successfully applied to the study of other closely related objects, such as representations of semigroups, representations of associative and Lie algebras, group actions on rings, and linear automata. Among these related fields, one should especially mention the rapidly expanding theory of varieties of representations of Lie algebras, presented in the recent book of Razmyslov [84].

The theory of varieties of group representations has been developing steadily since its inception, mainly in the USSR. During the first ten years about eighty papers were published in the field, including two important surveys of Plotkin [76, 77]. The results of this period were, to a certain extent, summarized in a monograph of Plotkin and the author [80], published in 1983 and until now the only detailed exposition of the subject.

This monograph was mostly devoted to the general framework of the theory which by the beginning of the 1980's was well developed and stable. At the same time, a number of more specialized and advanced topics were not covered in [80].

In the present notes several of these topics are considered. In no way does this book pretend to be a comprehensive treatment of the subject (as is clear enough from its title). Our aim is to familiarize the reader with the current state of knowledge in the areas we treat, to establish a number of interesting results, and to attract attention to several promising problems. The material presented here is quite recent: the greater part of the results have appeared within the last four years. Of course, the choice of material reflects the personal taste of the author.

The book starts with a preparatory Chapter 0 in which we collect all the necessary definitions, notation and facts from the theory of varieties of group representations. As a result, one can read this text independently of [80]. The reader familiar with the foundations of the theory can skip this chapter without any harm.

The main body of the book consists of four chapters. Chapter 1 deals with stable varieties. These varieties play the role analogous to that of nilpotent varieties of groups and linear algebras, and are among the most investigated in the field. A large part of the chapter is concerned with the important property of homogeneity, which goes back to Mal'cev [61]. In particular, one of the principal results states that over a field of characteristic zero there exists a canonical one-to-one correspondence between all varieties of associative algebras and the so-called homogeneous Magnus varieties of group representations, under which n-nilpotent varieties of algebras and n-stable (homogeneous) varieties of representations correspond to each other. This correspondence makes it possible to involve the well developed theory of varieties of algebras in the investigation of varieties of group representations. The main technical tool is the embedding of a free group into the algebra of formal power series, discovered by Magnus [57]. Somewhat isolated in the first chapter is § 1.6, where the main role belongs to connections with Lie algebras. Using Zel'manov's theorem [105] on the nilpotency of Lie algebras with the Engel identity, we show there that over a field of characteristic zero every unipotent variety is stable. In other words, the identity $(x-1)^n$ implies the identity $(x_1-1)(x_2-1)\dots(x_N-1)$ for some $N=N(n)$.

Chapter 2 is concerned with locally finite and locally finite-dimensional varieties. Such varieties are generated by their finite and, respectively, finite-dimensional representations, and this fact results in the character of the techniques used: periodic matrix groups, critical representations, irreducible representations of finite groups, etc. Generally speaking, the questions discussed here are rather traditional for variety theory, and the material of the chapter was developed under the strong influence of the theory of group varieties, in particular, of the excellent book of Hanna Neumann [68]. Among the results of Chapter 2, one can mention a nice and somewhat unexpected characterization of locally finite and locally finite-dimensional varieties, presented in § 2.2.

Chapter 3 is entirely devoted to the finite basis problem. The varieties considered there are locally finite or locally finite-dimensional, so that this chapter is a natural continuation of the previous one. Its main result is a theorem of the author and Nguyen Hung Shon [101] asserting that every stable-by-finite representation has a finite basis for its identities. It implies, in particular, that every representation of a finite group is finitely based. The proof uses to the full extent the machinery of critical objects and Cross varieties, as developed in papers of Oates–Powell [69] and Kovács–Newman [44], and a number of other technical tools specific to group representations.

Our final Chapter 4 does not focus on a single topic. Rather, it provides a selection of mutually independent results, including the most recent, which are of interest in their own right. Sections 4.1–4.2 are devoted to the finite basis problem for varieties satisfying multilinear identities, and to some natural connections between multilinear identities of group representations and those of associative algebras. In particular, using these connections and Kemer's solution of the Specht Problem [40], we deduce the following, *a priori* far from evident, fact: every system of multilinear identities of group representations over a field of characteristic zero is finitely based. Several other results show that if a variety satisfies a certain multilinear identity, then it is finitely based. Sections 4.3–4.4 deal with pure varieties over integral domains. Among the results of these sections, we mention an interesting theorem of G. M. Bergman on the product of ideals in free group rings which, in particular, gives an alternative proof of an earlier result of the author [94]: the product of pure varieties over a Dedekind domain is pure. An example outlined at the end of § 4.3 shows that this result cannot be generalized to arbitrary integral domains. Finally, the purpose of § 4.5 is to demonstrate that the theory of varieties of group representations has feedback to other fields of algebra. In this section we present a brief overview of some applications of our theory to varieties of groups, varieties of rings, and dimension subgroups.

For more detailed information concerning the contents of the book, the reader is referred to the introductory remarks at the beginning of each chapter.

One of the appealing features of the theory of varieties of group representations is the abundance and diversity of open problems. Some of these problems are mentioned in the present work, while many other can be found in [80]. Another attraction of the field is the broad range of techniques used. Here one can meet methods of traditional representation theory and the Fox free differential calculus, polynomial identities of rings and critical representations, the Magnus theory of the free group and connections with Lie algebras, free ideal rings and the classical Burnside theorems on matrix groups. Therefore we hope that the field treated in this book will be of interest to specialists in various branches of algebra.

We have tried to make this book accessible to a broad spectrum of readers including graduate students. A good graduate course in algebra (at the level of, say, Jacobson's *Basic Algebra*) plus some knowledge of variety theory should provide the necessary background. The exposition is, as a rule, detailed and complete, although, as in any more or less advanced course, it was impossible to make it absolutely self-contained. Sometimes we use without proofs certain facts from the theory of groups, rings, modules, etc., but in such cases precise references to the literature are always provided.

Despite its modest size, this book has had a difficult history. I worked on it during a very strained period in my life, and it is unlikely that I would have overcome the numerous obstacles without the help of many friends and colleagues. In the first place I am deeply indebted to Gregory Cherlin, Richard Lyons, Matthew Miller, Simon Thomas and Robert Wilson, who carefully read various portions of the manuscript and made numerous corrections and improvements: grammatical, stylistic and sometimes even mathematical. I am grateful to George Bergman for valuable and stimulating conversations in connection with § 4.3; to David Rohrlich and Eugene Speer for their help during my struggles with English grammar and various versions of TEX; and to the referee for an interesting and useful report. Finally, I would like to express my appreciation to Arkadiĭ Slin'ko and especially to Earl Taft, who helped me to recover the first draft of the manuscript after I had been separated from it for a long time.

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New Brunswick, New Jersey
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