Chapter 0

PRELIMINARIES

This introductory chapter provides a brief survey of basic concepts and facts from the theory of varieties of group representations which will be necessary for reading the main body of the book. The presentation is concise but, in general, self-contained: only a few standard and routine proofs are omitted. For a more detailed exposition of the foundations of the theory the interested reader is referred to Chapter 1 of [80].

0.1. Representations

Let K be an arbitrary but fixed commutative ring with unit which will usually be referred to as the *ground ring*. Consider a linear representation of a group G on a (left unitary) K-module V, that is, a group homomorphism $\rho : G \to \mathrm{Aut}_K V$. We suppose that elements of $\mathrm{Aut}_K V$ act on V on the right; then, denoting for any $v \in V$, $g \in G$

$$v \circ g = v\ \rho(g),$$

we obtain an *action* of G on V, i.e. a map $(v, g) \mapsto v \circ g$ from $V \times G$ to V satisfying the following conditions:

(i) for every $g \in G$ the map $v \mapsto v \circ g$ is an automorphism of V;

(ii) $(v \circ g_1) \circ g_2 = v \circ (g_1 g_2)$ for every $v \in V$, $g_1, g_2 \in G$.

Conversely, suppose there is given an action $\circ$ of a group G on a K-module V. Denoting for every $g \in G$ the map $v \mapsto v \circ g$ by $\rho(g)$, we evidently obtain a representation $\rho : G \to \mathrm{Aut}\, V$.

Thus from the standpoint of *many-sorted* universal algebra, a representation of a group over K can be treated as a two-sorted algebraic structure $(V, G, \circ)$, where V is a K-module, G a group, and $\circ$ an action of G on V. We will not adopt this standpoint in the present book, although it might be useful to keep it in mind, at least with respect to possible analogies, perspectives and generalizations.

Throughout these notes, a representation $\rho : G \to \operatorname{Aut} V$ is usually denoted by $\rho = (V, G)$; V is the *module* and G is the *acting group* of ρ. If $\rho = (V, G)$ is a representation, then its *kernel* $\operatorname{Ker}\rho = \operatorname{Ker}(V, G)$ is the kernel of the corresponding homomorphism $\rho : G \to \operatorname{Aut} V$; ρ is called *faithful* if $\operatorname{Ker}\rho = \{1\}$. In this case G can be considered as a subgroup of $\operatorname{Aut} V$. A representation $\rho = (V, G)$ is called *trivial* , or a *unit* representation, if $\operatorname{Ker}\rho = G$. If $V = \{0\}$, then ρ is said to be a *zero* representation.

In general, let $\rho = (V, G)$ be an arbitrary representation and $H = \operatorname{Ker}\rho$. Then the group $G/H = \bar{G}$ acts on V by the rule $v \circ gH = v \circ g$. We obtain a representation $\bar{\rho} = (V, \bar{G})$ which is certainly faithful; it is called the *faithful image* of ρ .

The group algebra of G over K is denoted by KG. The group G acts on KG by right multiplication giving the (right) *regular representation* (KG, G) which is denoted by $\operatorname{Reg}_K G$. If $\rho = (V, G)$ is any representation, then the action of G on V induces the action of KG on V by the natural rule: for arbitrary $v \in V$ and $u = \sum \lambda_i g_i \in KG$

$$v \circ u = \sum \lambda_i (v \circ g_i).$$

Therefore V can be regarded as a right KG-module (when it is clear from the context, we often say "G-module" instead of "KG-module"). Conversely, a KG-module structure on V determines an action of G on V, i.e. a representation (V, G).

Let $\rho = (V, G)$ and $\sigma = (W, H)$ be arbitrary representations over K. A *homomorphism* $\mu : \rho \to \sigma$ is a pair consisting of a homomorphism of K-modules $\mu : V \to W$ and a homomorphism of groups $\mu : G \to H$ (it is convenient to denote both these maps by a single letter) such that

$$\forall v \in V,\ g \in G : \ (v \circ g)^{\mu} = v^{\mu} \circ g^{\mu}. \tag{1}$$

The class of all group representations over K together with all homomorphisms forms a category denoted by REP-K. It is easy to verify that a homomorphism $\mu : (V,G) \to (W,H)$ is a monomorphism (epimorphism, isomorphism) in REP-K if and only if both $\mu : V \to W$ and $\mu : G \to H$ are monomorphisms (epimorphisms, isomorphisms). If $\rho = (V,G)$ is a representation, H a subgroup of G and W an H-submodule of V, then there naturally arises a representation (W,H) called a *subrepresentation* of ρ. Clearly σ is a subrepresentation of ρ if and only if there exists a monomorphism $\sigma \to \rho$.

Let $\mu : (V,G) \to (W,H)$ be a homomorphism and let $V_0 = \mathrm{Ker}(V \to W)$, $G_0 = \mathrm{Ker}(G \to H)$. It is easy to see that (V_0, G_0) is subrepresentation of (V,G), and that the following conditions are satisfied:

(i) $G_0 \triangleleft G$;

(ii) V_0 is a G_0-submodule of V;

(iii) the induced action of G_0 on V/V_0 is trivial.

The subrepresentation (V_0, G_0) is said to be the *kernel* of the homomorphism μ and is denoted by $\mathrm{Ker}\,\mu$.

On the other hand, let $\rho_0 = (V_0, G_0)$ be a subrepresentation of $\rho = (V,G)$ satisfying (i)–(iii). Then the group G/G_0 acts on the module V/V_0 by the rule

$$(v + V_0) \circ (gG_0) = v \circ g + V_0,$$

and we obtain a *factor-representation* $\rho/\rho_0 = (V/V_0, G/G_0)$. There exists a canonical epimorphism $\kappa : \rho \to \rho/\rho_0$ whose kernel is ρ_0, and usual arguments show that every epimorphic image of ρ can be realized in such a way.

A homomorphism $\mu : (V,G) \to (W,H)$ is called *right* if $\mu : V \to W$ is an isomorphism. Up to isomorphism, we may assume that a right homomorphism acts identically on the left side of the representation. For example, the canonical epimorphism of a representation $\rho = (V,G)$ on its faithful image $\bar{\rho} = (V,\ G/\mathrm{Ker}\,\rho)$ is a right epimorphism. Furthermore, it is clear that every right epimorphic image of ρ is isomorphic to some factor-representation $(V, G/H)$ where $H \subseteq \mathrm{Ker}\,\rho$. Hence the faithful image of any representation is its "smallest" right epimorphic image.

Two representations are said to be *equivalent* if their faithful images are isomorphic. The fact that representations ρ and σ are isomorphic or equivalent is denoted by $\rho \simeq \sigma$ or $\rho \sim \sigma$ respectively. In this notation

$$\rho \sim \sigma \iff \bar{\rho} \simeq \bar{\sigma}.$$

Note. In the classical theory of group representations, two representations (V, G) and (W, G) of a *fixed* group G are called equivalent if there exists an isomorphism $\mu : V \to W$ such that $(v \circ g)^\mu = v^\mu \circ g$ for all $v \in V, g \in G$ (cf. (1)) . In the category REP-K of representations of *arbitrary* groups this notion becomes a particular case of isomorphism, and is not very useful. The definition of equivalent representations adopted in these notes is motivated by the following observation: any two representations with isomorphic faithful images originate from the same faithful representation, i.e. *from the same action.* Therefore, as far as one is concerned with abstract properties of group actions, two representations with isomorphic faithful images should be treated as "equivalent" in some natural sense.

Let $\rho_i = (V_i, G_i)$, $i \in I$, be arbitrary representations. Denote by $V = \overline{\prod} V_i$ the Cartesian product of the K-modules V_i and by $G = \overline{\prod} G_i$ the Cartesian product of the groups G_i. Then G acts on V componentwise and so there arises a representation (V, G) which is called the *Cartesian product* of the representations ρ_i and is denoted by $\overline{\prod}\rho_i$. It is easy to see that ρ is the product of the objects ρ_i in the category REP-K. On the other hand, if we take the (restricted!) direct sum of modules $\bigoplus V_i$ and the direct product of groups $\prod G_i$, then the naturally arising representation $(\bigoplus V_i, \prod G_i)$ is called the *direct product* of the ρ_i's and is denoted by $\prod \rho_i$.

An *operation* on classes of group representations is a function U assigning to every class $\mathcal{X}$ of representations a class $\mathrm{U}\mathcal{X}$ such that

$$\mathcal{X} \subseteq \mathrm{U}\mathcal{X} \subseteq \mathrm{U}\mathcal{Y}$$

whenever $\mathcal{X} \subseteq \mathcal{Y}$. The *product* of operations is defined by the natural rule $(\mathrm{UV})\mathcal{X} = \mathrm{U}(\mathrm{V}\mathcal{X})$. An operation U is called a *closure operation* if $\mathrm{U}^2 = \mathrm{U}$; in this case the class $\mathrm{U}\mathcal{X}$ is U-closed for every $\mathcal{X}$, that is $\mathrm{U}(\mathrm{U}\mathcal{X}) = \mathrm{U}\mathcal{X}$.

From now on we fix the notation of several closure operations. Namely, if $\mathcal{X}$ is an arbitrary class of representations, then:

S$\mathcal{X}$ is the class of all subrepresentations of $\mathcal{X}$-representations (i.e. of representations from $\mathcal{X}$);

Q$\mathcal{X}$ is the class of all homomorphic images of $\mathcal{X}$-representations;

C$\mathcal{X}$ is the class of Cartesian products of $\mathcal{X}$-representations;

D$\mathcal{X}$ (D$_0\mathcal{X}$) is the class of direct products (of a finite number) of $\mathcal{X}$-representations;

V$\mathcal{X}$ is the class of all representations ρ such that there exists a right epimorphic image of ρ belonging to $\mathcal{X}$.

0.2. Identities and varieties

THE MAIN CONCEPTS AND EXAMPLES. Let F be the absolutely free group of countable rank with free generators $x_1, x_2, \ldots,$ KF its group algebra over the ground ring K, and $u(x_1, \ldots, x_n) = \sum \lambda_i f_i(x_1, \ldots, x_n)$ an element of KF. Suppose there is given a representation $\rho = (V, G)$ over K. If $g_1, \ldots, g_n \in G$ then $u(g_1, \ldots, g_n)$, being an element of the group algebra KG, acts naturally on V. We say that the formula

$$y \circ u(x_1, \ldots, x_n) \equiv 0$$

is an *identity* of the representation ρ if

$$v \circ u(g_1, \ldots, g_n) = 0$$

for any $v \in V$ and $g_1, \ldots, g_n \in G$. For brevity, the element $u = u(x_1, \ldots, x_n)$ is also said to be an identity of ρ. In other words, $u \in KF$ is an identity of $\rho = (V, G)$ if, for arbitrary $g_1, \ldots, g_n \in G$,

$$u(\rho(g_1), \ldots, \rho(g_n)) = 0$$

in $\mathrm{End}_K V$. If $\mathcal{X}$ is a class of representations, then $u \in KF$ is called an identity of $\mathcal{X}$ if it is an identity of every representation from $\mathcal{X}$.

0.2.1. Definition. *A class of group representations is called a* variety *if it consists of all representations satisfying a certain set of identities.*

A two-sided ideal of the group algebra KF is said to be *fully invariant* (or *completely invariant*, or *verbal*), if it is invariant under all endomorphisms of KF induced by endomorphisms of the group F. Note that a fully invariant ideal need not be invariant under all endomorphisms of the K-algebra KF. For example, let Δ be the *augmentation ideal* of KF, that is, the ideal generated by all elements $f - 1$, $f \in F$. Clearly it is fully invariant. On the other hand, Δ is not invariant under any endomorphism of KF taking a free generator x_i to an invertible element $\lambda \neq 1$ of K.

The significance of fully invariant ideals for our theory is illustrated by the following theorem. For every class of representations $\mathcal{X}$ denote by $\mathcal{X}^\alpha$ the set of all its identities in KF. Conversely, for every subset U of KF denote by U^β the class of all representations satisfying the identities from this subset.

0.2.2. Theorem. *The maps α and β determine a Galois correspondence between classes of group representations and subsets of KF. The closed elements under this correspondence are precisely the varieties of group representations over K and the fully invariant ideals of KF.*

Proof. It is evident that the maps α and β satisfy the following conditions:

(i) $\mathcal{X}_1 \subseteq \mathcal{X}_2 \Longrightarrow \mathcal{X}_1^\alpha \supseteq \mathcal{X}_2^\alpha$, $\quad U_1 \subseteq U_2 \Longrightarrow U_1^\beta \supseteq U_2^\beta$;

(ii) $\mathcal{X}^{\alpha\beta} \supseteq \mathcal{X}$, $\quad U^{\beta\alpha} \supseteq U$.

This exactly means that the maps α and β determine a Galois correspondence (see for example [12, Ch.2, §1]). From general properties of Galois correspondences, it follows that a class of representations $\mathcal{X}$ is closed (i.e. $\mathcal{X} = \mathcal{X}^{\alpha\beta}$) if and only if $\mathcal{X} = U^\beta$ for some $U \subseteq KF$. Similarly, a subset U of KF is closed (i.e. $U = U^{\beta\alpha}$) if and only if $U = \mathcal{X}^\alpha$ for some class of representations $\mathcal{X}$. Furthermore, if we restrict the maps α and β to the systems of closed elements, they will be one-to-one and mutually inverse.

We now prove the second assertion of the theorem. By definition, a class of representations $\mathcal{X}$ is a variety if and only if $\mathcal{X} = U^\beta$ for some $U \subseteq KF$. Hence varieties and closed classes are just the same.

Let U be a closed subset in KF. Then $U = \mathcal{X}^\alpha$ for some class $\mathcal{X}$, that is, U is the set of all identities of $\mathcal{X}$. It is easy to see that such a set must be a fully invariant ideal of KF. Conversely, suppose I is a fully invariant ideal of KF and prove that it is closed, that is $I^{\beta\alpha} = I$. Consider first the regular representation $\operatorname{Reg} F = (KF, F)$ of F and its factor-representation $\phi = (KF/I, F)$. We will show that the set of all identities of the latter coincides with I.

Let $u(x_1, \ldots, x_n) \in I$. Since I is fully invariant, for any $f_1, \ldots, f_n \in F$ we have $u(f_1, \ldots, f_n) \in I$, and so $u(f_1, \ldots, f_n)$ annihilates the module KF/I. Hence $u(x_1, \ldots, x_n)$ is an identity of the representation ϕ. On the other hand, let $u(x_1, \ldots, x_n)$ be an identity of $\phi = (KF/I, F)$. Take in KF/I the element $1 + I$; since

$$0 = (1 + I) \circ u(x_1, \ldots, x_n) = u(x_1, \ldots, x_n) + I,$$

we see that $u(x_1, \ldots, x_n)$ must belong to I.

In particular, we have $\phi \in I^\beta$. Let now $u \in I^{\beta\alpha}$. This means that u is an identity of the class I^β, so it is satisfied in ϕ. By the above, $u \in I$. Thus $I = I^{\beta\alpha}$. □

For any variety $\mathcal{X}$ of group representations, the ideal $\mathcal{X}^\alpha$ of its identities is denoted by $\operatorname{Id}\mathcal{X}$. By Theorem 0.2.2, the map $\mathcal{X} \mapsto \operatorname{Id}\mathcal{X}$ is a bijection between the varieties of group representations over K and the fully invariant ideals of KF. The set of varieties of group representations over K is denoted by $\mathbb{M}(K)$. In view of the preceding remark, the behavior of this set is controlled by the free group algebra KF.

Examples. 1. The class $\mathcal{S}$ of all trivial representations (recall that a representation $\rho = (V, G)$ is called trivial if each $g \in G$ acts identically on V) is a variety, for it can be determined by a single identity

$$y \circ (x - 1) \equiv 0.$$

It is easy to see that the ideal $\operatorname{Id}\mathcal{S}$ of identities of $\mathcal{S}$ is precisely the augmentation ideal Δ of KF.

2. A representation $\rho = (V, G)$ is called *stable of class n*, or simply *n-stable* (this terminology goes back to Kaloujnine [38]) if there is a series of G-modules

$$0 = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n = V$$

such that G acts trivially on every factor A_{i+1}/A_i.[1] A typical example of an n-stable representation is the representation $\mathrm{ut}_n(K) = (K^n, \mathrm{UT}_n(K))$ where K^n is the free K-module of rank n and $\mathrm{UT}_n(K)$ is the unitriangular matrix group of degree n over K acting on K^n in the natural way. The class $\mathcal{S}^n$ of all n-stable representations is a variety because it is determined by the identity

$$y \circ (x_1 - 1)(x_2 - 1)\dots(x_n - 1) \equiv 0 .$$

A straightforward verification shows that $\mathrm{Id}(\mathcal{S}^n) = \Delta^n$.

3. A representation is said to be *n-unipotent* if it satisfies the identity

$$y \circ (x - 1)^n \equiv 0.$$

The variety of all n-unipotent representations is denoted by $\mathcal{U}_n$. Evidently $\mathcal{S}^n \subseteq \mathcal{U}_n$. On the other hand, suppose that K is a field, then a classical theorem of Kolchin [42] states that if a *finite-dimensional* representation $\rho = (V, G)$ is unipotent, then it is stable (there is no need to speak here about the class of stability and the class of unipotency because they both can be chosen to coincide with $\dim V$). In other words, in the finite-dimensional case stable representations and unipotent representations are exactly the same.

The Kolchin Theorem led naturally to the problem of whether every (not necessarily finite-dimensional) unipotent representation over a field is stable, i.e. whether $\mathcal{U}_n \subseteq \mathcal{S}^N$ for some $N = N(n)$. If the ground field K is of prime characteristic, a negative answer can be obtained immediately. But for $\mathrm{char}\, K = 0$ the problem has remained unsolved for about 35 years. We will return to this question in § 1.6.

4. For any variety of groups Θ denote by $\omega\Theta$ the class of all representations $\rho = (V, G)$ such that $G/\mathrm{Ker}\,\rho \in \Theta$. This class is a variety because if Θ is determined by a set of group identities $\{f_i\}$, then $\omega\Theta$ is determined by the set $\{f_i - 1\}$. Note that the map $\Theta \mapsto \omega\Theta$ is injective, for if $G \in \Theta_1 \setminus \Theta_2$, then it is clear that $\mathrm{Reg}\, G = (KG, G) \in \omega\Theta_1 \setminus \omega\Theta_2$. Thus there exists a

[1] The word "stable" is overused in today's mathematics and, probably, is not optimal here. But this term is already quite common in the field, so we decided not to change it.

natural embedding of the set of varieties of groups into the set $\mathbb{M}(K)$. The varieties of group representations of the form $\omega\Theta$ will be sometimes called the *varieties of group type.*

It is not hard to identify the ideal $\mathrm{Id}(\omega\Theta)$ of identities of $\omega\Theta$. It is generated, as a right ideal, by all $f-1$, where f belongs to the Θ-verbal subgroup $\Theta(F)$ of F. Equivalently, $\mathrm{Id}(\omega\Theta)$ is the kernel of the natural epimorphism $KF \to K[F/\Theta(F)]$.

5. This example complements the preceding one and demonstrates that there is an essential difference between identities of abstract groups and identities of their representations. Take the special linear group $\mathrm{SL}_2(K)$ over a field K of characteristic zero. Since $\mathrm{SL}_2(K)$ contains free nonabelian subgroups, it has no nontrivial group identities. Therefore from the standpoint of group identities the classical group $\mathrm{SL}_2(K)$ is not an "interesting" object. Consider now another classical object: the canonical two-dimensional representation $\mathrm{sl}_2(K) = (K^2, \mathrm{SL}_2(K))$. This representation has many interesting identities, for example, an elementary verification shows that

$$(x_1 + x_1^{-1})x_2 - x_2(x_1 + x_1^{-1}) \tag{1}$$

is an identity of $\mathrm{sl}_2(K)$.[2]

A similar phenomenon is valid for other classical matrix groups over an infinite field: as abstract groups, they usually have no nontrivial identities, while their canonical representations certainly do (for instance, by the Amitsur–Levitzki Theorem, every n-dimensional representation satisfies the so-called standard polynomial identity of degree $2n$).

6. Evidently the class $\mathcal{O}$ of all representations over K and the class $\mathcal{E}$ of all zero representations are varieties; they are called *trivial* varieties. The corresponding fully invariant ideals in KF are $\{0\}$ and KF respectively.

0.2.3. Proposition. *If K is a field, then every proper (i.e. $\neq KF$) fully invariant ideal of KF is contained in the augmentation ideal Δ.*

[2] Moreover, it was proved in [51] that every identity of $\mathrm{sl}_2(K)$ is a consequence of (1).

Proof. Let I be such an ideal and $u(x_1, \ldots, x_n) = \sum \lambda_i f_i(x_1, \ldots, x_n)$ an element from I. Since I is completely invariant,

$$u(1, \ldots, 1) = \sum \lambda_i f_i(1, \ldots, 1) = \sum \lambda_i \in I.$$

If $\sum \lambda_i \neq 0$, then $1 \in I$, that is $I = KF$, which is impossible. Hence $\sum \lambda_i = 0$, and so $I \subseteq \Delta$. □

Equivalently: *Every nonzero (i.e. $\neq \mathcal{E}$) variety of group representations over a field contains $\mathcal{S}$.*

Let $\mathcal{X}$ be a variety and $\rho = (V, G)$ an arbitrary representation. It is easy to see that if A_i $(i \in I)$ are G-submodules of V such that the corresponding factor-representations $(V/A_i, G)$ belong to $\mathcal{X}$, then $(V/\cap A_i,\ G)$ belongs to $\mathcal{X}$ as well. Therefore there exists the smallest G-submodule A of V such that $(V/A, G) \in \mathcal{X}$. This submodule is called the $\mathcal{X}$*-verbal* of ρ and is denoted by $\mathcal{X}^*(\rho) = \mathcal{X}^*(V, G)$.

0.2.4. Lemma. *Let $\mathcal{X}$ be a variety. Then the following assertions are valid:*

(i) *If $\rho = \prod \rho_i$, then $\mathcal{X}^*(\rho) = \prod \mathcal{X}^*(\rho_i)$.*
(ii) *If $\rho \subseteq \sigma$, then $\mathcal{X}^*(\rho) \subseteq \mathcal{X}^*(\sigma)$.*
(iii) *If $\mu : \rho \to \sigma$ is a homomorphism, then $\mathcal{X}^*(\rho^\mu) = (\mathcal{X}^*(\rho))^\mu$.*

Proof is routine. For example, let us prove the last assertion. Recall that two representations are said to be equivalent (see Section 0.1) if their faithful images are isomorphic. It is evident that every representation has the same identities as its faithful image, and therefore if a representation ϕ belongs to a variety $\mathcal{X}$, then all representations equivalent to ϕ belong to $\mathcal{X}$ as well.

Now let $\rho = (V, G)$, $\sigma = (W, H)$, and let $\mu : \rho \to \sigma$ be an epimorphism. Denote $\mathcal{X}^*(\rho) = A$ and $\mathcal{X}^*(\sigma) = B$. We will show that $A^\mu = B$. Note first that the epimorphism μ induces an epimorphism of factor-representations

$$(V/A, G) \to (W/A^\mu, H).$$