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Progress Report on the Telescope Conjecture

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The Telescope Conjecture (made public in a lecture at Northwestern University in 1977) says that the v_n–periodic homotopy of a finite complex of type n has a nice algebraic description. It also gives an explicit description of certain Bousfield localizations. In this paper we outline a proof that it is *false* for $n = 2$ and $p \geq 5$. A more detailed account of this work will appear in [Rav]. In view of this result, there is no longer any reason to think it is true for larger values of n or smaller primes p.

In Section 1 we will give some background surrounding the conjecture. In Section 2 we outline Miller's proof of it for the case $n = 1$ and $p > 2$. This includes a discussion of the localized Adams spectral sequence. In Section 3 we describe the difficulties in generalizing Miller's proof to the case $n = 2$. We end that section by stating a theorem (3.5) about some differentials in the Adams spectral sequence, which we prove in Section 4. This material is new; I stated the theorem in my lecture at the conference, but said nothing about its proof. In Section 5 we construct the parametrized Adams spectral sequence, which gives us a way of interpolating between the Adams spectral sequence and the Adams–Novikov spectral sequence. We need this new machinery to use Theorem 3.5 to disprove the Telescope Conjecture. This argument is sketched in Section 6.

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1 Background

Recall that for each prime p there are generalized homology theories $K(n)_*$ (the Morava K–theories) for each integer $n \geq 0$ with the following properties:

(i) $K(0)_*$ is rational homology and $K(1)_*$ is one of $p-1$ isomorphic summands of mod p complex K–theory.

(ii) For $n > 0$, $K(n)_*(\text{pt.}) = \mathbf{Z}/(p)[v_n, v_n^{-1}]$ with $|v_n| = 2p^n - 2$.

(iii) There is a Künneth isomorphism

$$K(n)_*(X \times Y) \cong K(n)_*(X) \otimes K(n)_*(Y).$$

(iv) If X is a finite spectrum with $K(n)_*(X) = 0$, then $K(n-1)_*(X) = 0$.

(v) If the p–localization of X (as above) is not contractible, then

$$K(n)_*(X) \neq 0 \quad \text{for} \quad n \gg 0.$$

The last two properties imply that we can make the following.

Definition 1.1 *A noncontractible finite p–local spectrum X has* **type** *n if n is the smallest integer such that $K(n)_*(X) \neq 0$.*

Definition 1.2 *If X as above has type n then a v_n–***map** *on X is a map*

$$\Sigma^d X \xrightarrow{f} X$$

with $K(n)_*(f)$ an isomorphism and $K(m)_*(f) = 0$ for all $m \neq n$.

The Periodicity Theorem of Hopkins–Smith [HS] says that such a map always exists and is unique in the sense that if g is another such map then some iterate of f is homotopic to some iterate of g. The Telescope Conjecture concerns the telescope $\widehat{X}$, which is defined to be the homotopy direct limit of the system

$$X \xrightarrow{f} \Sigma^{-d} X \xrightarrow{f} \Sigma^{-2d} X \xrightarrow{f} \cdots$$

The Periodicity Theorem tells us that this is independent of the choice of the v_n–map f.

The motivation for studying $\widehat{X}$ is that the associated Adams–Novikov spectral sequence has nice properties. We will illustrate with some simple examples. Suppose

$$BP_*(X) = BP_*/I_n = BP_*/(p, v_1, \cdots v_{n-1}).$$

This happens when X is the Toda complex $V(n-1)$. These are known to exist for small n and large p. Then

$$BP_*(\widehat{X}) = v_n^{-1}BP_*/I_n.$$

The E_2–term of the associated Adams–Novikov spectral sequence is

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{BP_*(BP)}(BP_*, v_n^{-1}BP_*/I_n),$$

which can be computed directly. For more details, see 5.1.14 and Chapter 6 of [Rav86]. It is a free module over $K(n)_*$. In particular when $n = 2$ and $p \geq 5$ (in which case the spectrum $V(1)$ is known to exist) it has total (for all values of s) rank 12 and vanishes for $s > 4$. This means that the Adams–Novikov spectral sequence collapses and there are no extension problems.

The computability of this Ext group was one of the original motivations for studying v_n–periodic homotopy theory.

However, we do not know that this Adams–Novikov spectral sequence converges to $\pi_*(\widehat{X})$. It is known [Rav87] to converge to $\pi_*(L_nX)$, where L_nX denotes the Bousfield localization of X with respect to $E(n)$–theory. (When X is a finite spectrum of type n, this is the same as the localization with respect to $K(n)$–theory.) Since $\widehat{X}$ is $K(n)_*$–equivalent to X, there are maps

$$X \xrightarrow{i} \widehat{X} \xrightarrow{\lambda} L_nX.$$

The Telescope Conjecture says that λ is an equivalence, or equivalently that the Adams–Novikov spectral sequence converges to $\pi_*(\widehat{X})$. This statement is trivial for $n = 0$, known to be true for $n = 1$ ([Mil81] and [Mah82]). The object of this paper is to sketch a counterexample for $n = 2$ and $p \geq 5$.

2 Miller's proof for $n = 1$ and $p > 2$

It is more or less a formality to reduce the Telescope Conjecture for a given value of n and p to proving it for one particular p–local finite spectrum of type n. We will outline Miller's proof for the mod p Moore spectrum $V(0)$. In that case the v_1–map

$$\Sigma^{2p-2}V(0) \xrightarrow{\alpha} V(0) \tag{2.1}$$

is the map discovered long ago by Adams in [Ada66]. There is a map

$$S^{2p-3} \longrightarrow S^0 \longrightarrow V(0)$$

which corresponds to an element in the Adams–Novikov spectral sequence called $h_{1,0}$. The Telescope Conjecture says that

$$\pi_*(\widehat{V(0)}) = K(1)_* \otimes E(h_{1,0}) \tag{2.2}$$

where $E(\cdot)$ denotes an exterior algebra.

Miller studies this problem by looking at the classical Adams spectral sequence for $\pi_*(V(0))$. In its E_2–term there is an element

$$v_1 \in E_2^{1,2p-1}$$

that corresponds to the Adams map α. One can formally invert this element and get a localized Adams spectral sequence converging to $\pi_*(\widehat{V(0)})$. (This convergence is not obvious, and is proved in [Mil81].)

We will describe the construction of this localized Adams spectral sequence. Recall that the classical Adams spectral sequence for the homotopy of spectrum X is constructed as follows. One has an *Adams resolution for* X, which is a diagram of the form

$$\begin{array}{ccccccc} X = X_0 & \longleftarrow & X_1 & \longleftarrow & X_2 & \longleftarrow & \cdots \\ {\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & \\ K_0 & & K_1 & & K_2 & & \end{array}$$

with the following properties.

(i) Each K_s is a wedge of suspensions of mod p Eilenberg–Mac Lane spectra.

(ii) Each map f_s induces a monomorphism in mod p homology.

(iii) X_{s+1} is the fibre of f_s.

The *canonical Adams resolution* for X is obtained by setting

$$K_s = X_s \wedge H/(p).$$

A map $g : X \to Y$ induces a map of Adams resolutions, i.e., a collection of maps $g_s : X_s \to Y_s$ with suitable properties. The map g has *Adams filtration* $\geq t$ if it lifts to a map $g' : X \to Y_t$. In this case it is automatic that g_s lifts to Y_{s+t}.

Now consider the example at hand, namely $X = V(0)$. The map α has Adams filtration 1, so we have maps

$$V(0) = X_0 \xrightarrow{\alpha_0'} \Sigma^{-q} X_1 \xrightarrow{\alpha_1'} \Sigma^{-2q} X_2 \xrightarrow{\alpha_2'} \cdots,$$

where $q = 2p - 2$. We define $\widehat{X}_s$ to be the limit of

$$X_s \xrightarrow{\alpha'_s} \Sigma^{-q} X_{s+1} \xrightarrow{\alpha'_{s+1}} \Sigma^{-2q} X_{s+2} \xrightarrow{\alpha'_{s+2}} \cdots,$$

and $\widehat{K}_s$ to be the cofibre of the map $\widehat{X}_{s+1} \to \widehat{X}_s$, or equivalently the limit of

$$K_s \longrightarrow \Sigma^{-q} K_{s+1} \longrightarrow \Sigma^{-2q} K_{s+2} \longrightarrow \cdots,$$

Like K_s, it is a bouquet of mod p Eilenberg–Mac Lane spectra. These spectra are defined for *all integers* s, not just for $s \geq 0$ as in the classical case.

Thus we get a *localized Adams resolution*, i.e., a diagram

$$\begin{array}{ccccccccc} \cdots & \longleftarrow & \widehat{X}_s & \longleftarrow & \widehat{X}_{s+1} & \longleftarrow & \widehat{X}_{s+2} & \longleftarrow & \cdots \\ & & {\scriptstyle \widehat{f}_s}\downarrow & & {\scriptstyle \widehat{f}_{s+1}}\downarrow & & {\scriptstyle \widehat{f}_{s+2}}\downarrow & & \\ & & \widehat{K}_s & & \widehat{K}_{s+1} & & \widehat{K}_{s+2} & & \end{array} \tag{2.3}$$

and a spectral sequence converging to the homotopy of the telescope $\widehat{V(0)}$, which is the limit of

$$\widehat{X}_0 \longrightarrow \widehat{X}_{-1} \longrightarrow \widehat{X}_{-2} \longrightarrow \cdots$$

To prove the spectral sequence converges, one must show that the inverse limit of the $\widehat{X}_s$ is contractible.

Unlike the classical Adams spectral sequence, which is confined to the first quadrant, the localized Adams spectral sequence is a full plane spectral sequence with $E_1^{s,t}$ conceivably nontrivial for all integers s and t. However, it can be shown that the E_2–term has a vanishing line of slope $1/q$, namely

$$E_2^{s,t} = 0 \quad \text{for} \quad s > \frac{t-s+1}{q}.$$

Fortunately the E_2–term of the localized Adams spectral sequence is far simpler than that of the usual Adams spectral sequence. In order to describe it we need to recall some facts about the Steenrod algebra A. Its dual is

$$A_* = E(\tau_0, \tau_1, \cdots) \otimes P(\xi_1, \xi_2, \cdots)$$

where $P(\cdot)$ denotes a polynomial algebra over $\mathbf{Z}/(p)$. We will denote these two factors by Q_* and P_* respectively.

We will use the homological (as opposed to cohomological) formulation of the Adams spectral sequence for $\pi_*(X)$, so the E_2–term is

$$\operatorname{Ext}_{A_*}(\mathbf{Z}/(p), H_*(X)) \tag{2.4}$$

where $H_*(X)$ (the mod p homology of X) is regarded as a comodule over A_*.

There is an extension of Hopf algebras

$$P_* \longrightarrow A_* \longrightarrow Q_*$$

which leads to a Cartan–Eilenberg spectral sequence converging to (2.4) with

$$E_2 = \mathrm{Ext}_{P_*}(\mathbf{Z}/(p), \mathrm{Ext}_{Q_*}(\mathbf{Z}/(p), H_*(X))).$$

The inner Ext group is easy to compute since Q_* is dual to an exterior algebra. For $X = V(0)$ it is

$$P(v_1, v_2, \cdots) \quad \text{with} \quad v_n \in \mathrm{Ext}^{1,2p^n-1}.$$

(The elements v_n correspond so closely to the generators of $\pi_*(BP)$ that we see no point in making a notational distinction between them.)

For odd primes the Cartan–Eilenberg spectral sequence collapses. (See [Rav86, 4.4.3]. It is stated there only for $X = S^0$, but the proof given will work for any X.) It follows that

$$\mathrm{Ext}^s_{A_*}(\mathbf{Z}/(p), H_*(X)) \cong \bigoplus_{i+j=s} \mathrm{Ext}^i_{P_*}(\mathbf{Z}/(p), \mathrm{Ext}^j_{Q_*}(\mathbf{Z}/(p), H_*(X))). \qquad (2.5)$$

We can pass to the telescope $\widehat{V(0)}$ by inverting v_1. Then we have the following very convenient change-of-rings isomorphism.

$$\begin{aligned} \mathrm{Ext}_{P_*}(\mathbf{Z}/(p), v_1^{-1}P(v_1, v_2, \cdots)) &\cong \mathrm{Ext}_{B(1)_*}(\mathbf{Z}/(p), K(1)_*) \qquad (2.6)\\ &\cong K(1)_* \otimes \mathrm{Ext}_{B(1)_*}(\mathbf{Z}/(p), \mathbf{Z}/(p)) \end{aligned}$$

where $K(1)_*$ as usual denotes the ring $v_1^{-1}P(v_1)$ and

$$B(1)_* = P(\xi_1, \xi_2, \cdots)/(\xi_i^p).$$

This Hopf algebra has a cocommutative coproduct, so its Ext group is easy to compute and we have

$$\mathrm{Ext}_{B(1)_*}(\mathbf{Z}/(p), \mathbf{Z}/(p)) \cong E(h_{1,0}, h_{2,0}, \cdots) \otimes P(b_{1,0}, b_{2,0}, \cdots)$$

where

$$\begin{aligned} h_{i,0} &\in \mathrm{Ext}^{1,2p^i-2}\\ b_{i,0} &\in \mathrm{Ext}^{2,2p^{i+1}-2p}. \end{aligned}$$

This should be compared with the localized form of the Adams–Novikov spectral sequence, in which the E_2-term is

$$\mathrm{Ext}_{BP_*(BP)}(BP_*, v_1^{-1}BP_*/(p)).$$

One can get a spectral sequence converging to this called the algebraic Novikov spectral sequence by filtering BP_* by powers of the ideal

$$I = (p, v_1, v_2, \cdots). \tag{2.7}$$

The E_2–term of this spectral sequence is a regraded form of (2.6). We denote the r^{th} differential in this spectral sequence by δ_r. These can all be computed by algebraic methods coming from BP–theory. In this case we have

$$\delta_2(h_{i+1,0}) = v_1 b_{i,0} \quad \text{for} \quad i > 0.$$

Miller uses this to deduce that there are similar differentials in the localized Adams spectral sequence, namely

$$d_2(h_{i+1,0}) = v_1 b_{i,0}.$$

This gives

$$E_3 = E_\infty = K(1)_* \otimes E(h_{1,0}),$$

which proves the Telescope Conjecture for $n = 1$ and $p > 2$.

3 Difficulties for $n = 2$

One can mimic Miller's argument for $n = 2$ and $p \geq 5$. In that case one has the spectrum

$$V(1) = S^0 \cup_p e^1 \cup_{\alpha_1} e^{2p-1} \cup_p e^{2p},$$

which is the cofibre of the Adams map α of (2.1). There is a v_2–map

$$\Sigma^{2p^2-2} V(1) \xrightarrow{\beta} V(1)$$

constructed by Larry Smith [Smi71] and H. Toda [Tod71]. The Adams E_2–term is

$$\mathrm{Ext}_{P_*}(\mathbf{Z}/(p), P(v_2, v_3, \cdots)).$$

We can use the map β to localize this Adams spectral sequence in the same way as Miller localized the one for $V(0)$. The resulting E_2–term is

$$K(2)_* \otimes \mathrm{Ext}_{B(2)_*}(\mathbf{Z}/(p), \mathbf{Z}/(p))$$

where

$$B(2)_* = P(\xi_1, \xi_2, \cdots)/(\xi_i^{p^2}).$$

This does not have a cocommutative coproduct, so its Ext group is not as easy to compute as (2.6), but it is still manageable. It is a subquotient of the cohomology of the cochain complex

$$C^{*,*} = E(h_{1,0}, h_{2,0}, \cdots; h_{1,1}, h_{2,1}, \cdots) \otimes P(b_{1,0}, b_{2,0}, \cdots; b_{1,1}, b_{2,1}, \cdots)$$

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where

$$\begin{aligned} h_{i,j} &\in C^{1,2p^j(p^i-1)} \\ b_{i,j} &\in C^{2,2p^{j+1}(p^i-1)} \end{aligned}$$

and the coboundary ∂ is given by

$$\begin{aligned} \partial(h_{i,0}) &= \pm h_{1,0}h_{i-1,1} \\ \partial(h_{i,1}) &= 0 \\ \partial(b_{i,0}) &= \pm h_{1,1}b_{i-1,1} \\ \partial(b_{i,1}) &= 0. \end{aligned} \tag{3.1}$$

There is also an algebraic Novikov spectral sequence with the following differentials.

$$\delta_2(h_{i,0}) = \pm v_2 b_{i-2,1} \tag{3.2}$$

$$\delta_{1+p^{i-1}}(h_{i,1}) = \pm v_2^{p^{i-1}} b_{i-2,0} \quad \text{for } i \geq 3 \tag{3.3}$$

The reader may object to (3.2) on the grounds that $h_{i,0}$ is not a cocycle in $C^{*,*}$, and he would be correct. It would be more accurate to say that the algebraic Novikov spectral sequence has differentials formally implied by (3.2), such as

$$\begin{aligned} \delta_2(h_{1,0}h_{i,0}) &= \pm v_2 h_{1,0} b_{i-2,1} \quad \text{and} \\ \delta_2(h_{i-1,1}h_{i,0}) &= \pm v_2 h_{i-1,1} b_{i-2,1}. \end{aligned}$$

In any case these differentials kill the elements $b_{i,j}$ and $h_{i+2,j}$ for all $i > 0$, and the E_2-term of the Adams–Novikov spectral sequence is the cohomology of

$$K(2)_* \otimes E(h_{1,0}, h_{1,1}, h_{2,0}, h_{2,1})$$

with the coboundary given by (3.1), namely

$$\partial(h_{2,0}) = \pm h_{1,0}h_{1,1}.$$

This is a $K(2)_*$-module of rank 12 with basis

$$E(h_{2,1}) \otimes \{1,\ h_{1,0},\ h_{1,1},\ h_{1,0}h_{2,0},\ h_{1,1}h_{2,0},\ h_{1,0}h_{1,1}h_{2,0}\}. \tag{3.4}$$

This is the value of $\pi_*(\widehat{V(1)})$ predicted by the Telescope Conjecture.

The difficulty is that while Miller's methods allow us to translate the algebraic differentials implied by (3.2) into differentials in the localized Adams spectral sequence, they do *not* enable us to do so for those of (3.3). The latter would give us d_r's for arbitrarily large r, and such differentials could be interfered with by other shorter differentials not related to the algebraic Novikov spectral sequence.

The following result says that such interfering differentials *do* occur in the localized Adams spectral sequence.

Theorem 3.5 *In the localized Adams spectral sequence for $\widehat{V(1)}$ for $p \geq 5$,*

$$d_{2p}(h_{i,1}) = \pm v_2 b_{i-1,0}^p \quad \text{for } i \geq 2$$

modulo nilpotent elements.

The proof of this will be sketched below in Section 4. For $i = 2$ this can be deduced from the Toda differential [Rav86, 4.4.22] by direct calculation.

This result shows that the E_{2p}-term of the localized Adams spectral sequence is a subquotient of

$$E(h_{1,0}, h_{1,1}, h_{2,0}) \otimes P(b_{1,0}, b_{2,0}, \cdots)/(b_{i,0})^p.$$

Even though this is infinite dimensional, it is too small in the sense that it appears to have only two elements with Novikov filtration one (namely $h_{1,0}$ and $h_{1,1}$), while there are three such elements in (3.4).

4 Computing the differentials $d_{2p}(h_{i,1})$

The purpose of this section is to prove Theorem 3.5. The following is rationale for these differentials, which will be made more rigorous and precise below. In the appropriate form of the cobar complex, we have

$$-d(h_{i+2,0}) \equiv h_{1,0}h_{i+1,1} + v_2 b_{i,1} \tag{4.1}$$

modulo terms with higher Adams filtration. It follows that the target of this differential must be a permanent cycle in the localized Adams spectral sequence. Now suppose we knew that

$$d_{2p-1}(b_{i,1}) = h_{1,0} b_{i,0}^p. \tag{4.2}$$

Then combining this with (4.1) would determine the differential on $h_{i+1,1}$, giving Theorem 3.5 up to suitable indeterminacy.

The Toda differential

For $i = 1$, (4.2) is the Toda differential, first established in [Tod67]. The following is a reformulation of Toda's proof. The generator of $\pi_{2p-2}(BU)$ is represented by a map which extends (via the loop space structure of BU) to a map

$$\Omega S^{2p-1} \longrightarrow BU,$$

which can be composed with the map

$$\Sigma\Omega^2 S^{2p-1} \longrightarrow \Omega S^{2p-1}$$

(adjoint to the identity map) to give a vector bundle over $\Sigma\Omega^2 S^{2p-1}$. Its Thom spectrum is the cofibre of a stable map

$$\Omega^2 S^{2p-1} \xrightarrow{f} S^0.$$

Now $\Omega^2 S^{2p-1}$ splits stably into an infinite wedge of finite spectra B_i for $i > 0$ described explicitly by Snaith in [Sna74]. Localization at p makes B_i contractible except when $i \equiv 0$ or $1 \bmod p$, and makes B_{pj+1} equivalent to $\Sigma^{2p-3} B_{pj}$ for $j > 0$. The best way to see this is to look at mod p homology. We have

$$H_*(\Omega^2 S^{2p-1}; \mathbf{Z}/(p)) = E(x_0, x_1, \cdots) \otimes P(y_1, y_2, \cdots)$$

where the dimensions of x_j and y_j are $2p^j(p-1)-1$ and $2p^j(p-1)-2$ respectively.

In order to describe the Snaith splitting homologically, it is convenient to assign a *weight* to each monomial. We do this by defining the weight of both x_j and y_j to be p^j. This leads to a direct sum decomposition of the homology corresponding to the Snaith splitting of the suspension spectrum, i.e., $H_*(B_i)$ is spanned by the monomials of weight i.

Now observe that the only generator whose weight is not divisible by p is x_0, which is an exterior generator. It follows that multiplication by x_0 gives an isomorphism from the subspace spanned by monomials with weight divisible by p to the that spanned by the ones with weight congruent to 1 mod p. This isomorphism can be realized by a p–local equivalence $\Sigma^{2p-3} B_{pi} \to B_{pi+1}$. Moreover, every monomial has weight congruent to 0 or 1 mod p.

Also note that the first monomial of weight pi is y_1^i, which has dimension $2i(p^2-p-1)$. It follows that

$$(B_{pi})_{(p)} = \Sigma^{2i(p^2-p-1)} D_i$$

for some (-1)–connected finite spectrum D_i.

Thus the Snaith splitting (after localizing at p) has the form

$$\Omega^2 S_+^{2p-1} \simeq (S^0 \vee S^{2p-3}) \wedge \bigvee_{i \geq 0} \Sigma^{2i(p^2-p-1)} D_i.$$