The Work of J. F. Adams[1]

I first met Frank here in Manchester in 1964, when this building was being planned. I remember from the first feeling that he was a far more impressive man than the anecdotes of his exploits had led me to expect, and a far nicer one. I also felt humbled by the sheer amount of mathematics that he knew and perhaps more so by the amount that he somehow assumed I knew. I feel a little the same way now, faced with this audience and this topic. Still, I don't want to spend much time in reminiscence.[2] I want rather to give a quick guided tour through Frank's work, largely letting it speak for itself.

I should say that Frank's collected works are to be published in the near future by the Cambridge University Press. Like this talk, the collected works are organized by subject matter rather than by strict chronology. However, I will begin not quite at the beginning of his work with a sequence of four papers submitted between 1955 and 1958. All dates cited are dates of submission, not necessarily of appearance.

A. The cobar construction, the Adams spectral sequence, higher order cohomology operations, and the Hopf invariant one problem

1. *On the chain algebra of a loop space* (1955, with Peter Hilton) [**5**][3]
2. *On the cobar construction* (1956) [**6**]

Let K be a CW-complex with trivial 1-skeleton. In the first paper, a DGA-algebra $A(K)$ is constructed whose homology is the Pontryagin algebra $H_*(\Omega K)$; as an algebra, $A(K)$ is free on generators in bijective correspondence with the cells of K (other than the vertex). As Kathryn Hess explained in her talk a few hours ago, this Adams-Hilton model is small enough to be of concrete value for computations and is still being used and studied today. In the second paper, a larger, but functorial, DGA is given whose homology is $H_*(\Omega K)$, namely the cobar construction $F(C_*(K))$. This construction was discussed in John McCleary's talk on Hochschild homology. Nowadays, an obvious and trivial next step after the introduction of

[1] Reconstruction and expansion of the talk given at the conference, most of which was not written out beforehand

[2] A more personal tribute has been published in The Mathematical Intelligencer, Vol. 12, No. 1, 1990, 40–48.

[3] Details of publication of Adams' works discussed here can be found in the complete bibliography which follows this paper.

the cobar construction would be to filter it and so arrive at what is called the Eilenberg-Moore spectral sequence for the computation of $H_*(\Omega K)$. In fact, Moore and Adams were already in contact before this paper was written, and it was cited by Eilenberg and Moore as an important precursor to their work.

3. *On the structure and applications of the Steenrod algebra* (1957) [**9**]

Adams viewed this paper as a step towards the solution of the Hopf invariant one problem. The main theorem states that if $\pi_{2n-1}(S^n)$ and $\pi_{4n-1}(S^{2n})$ both contain elements of Hopf invariant one, then $n \leq 4$. It is now chiefly celebrated for the introduction of the Adams spectral sequence converging from $\mathrm{Ext}_A^{s,t}(H^*(X), Z_p)$ to ${}_p\pi_*^s(X)$. Products are defined in the spectral sequence when $X = S^0$, and the sub-Hopf algebras A_r are used to compute products of the elements h_i inductively, where h_i corresponds to Sq^{2^i}. The basic argument runs as follows. Let $n = 2^m$, $m \geq 3$. Assuming that h_m is a permanent cycle, $h_0(h_m)^2$ would survive to E_∞ if $d_2 h_{m+1} = 0$. This would contradict the fact that, in $\pi_*^s(S^0)$, $2x^2 = 0$ if $\deg(x)$ is odd. This seems straightforward enough today, but it was revolutionary at the time. The idea of reducing such a fundamental topological problem as Hopf invariant one to the non-triviality of a particular differential in a spectral sequence was quite new and unexpected.

Adams was curiously modest about the Adams spectral sequence. He always referred to it as a formalization of the Cartan-Serre method of killing homotopy groups. I think we all see it as something very much more than that. Its introduction was a watershed, and it substantially raised the level of algebraic sophistication of our subject.

4. *On the non-existence of elements of Hopf invariant one* (1958) [**14**]

If $\pi_{2n-1}(S^n)$ contains an element of Hopf invariant one, then $n = 1, 2, 4$, or 8. The proof is based on showing that Sq^{2^m} decomposes in terms of secondary cohomology operations if $m > 3$. The paper contains definitive homological algebra for the study of Ext_A, including minimal resolutions and the cobar construction with its $\smile$ and $\smile_1$ products. It uses Milnor's description of A^* to redo the calculations in the previous paper. It gives a detailed study of stable secondary cohomology operations via universal examples, which are generalized two-stage Postnikov systems. The results include axioms for the operations, existence and uniqueness theorems, the relationship between the operations and Tor_A^2, and a Cartan formula. Particular operations are studied via homological algebra, and a key computation in $\mathbb{C}P^\infty$ is used to start the induction which shows that the undetermined constants in the decompositions of the Sq^{2^m} are non-zero.

Adams was a problem solver. He introduced exactly the tools he needed to solve the problems he studied, and he had relatively little interest in Bourbaki style analysis of the foundations or in systematic calculations. He had an extraordinary talent for proving important and easily formulated conceptual theorems through a mix of new ideas, new foundational constructions, and adroit calculations. The solution of the Hopf invariant one problem was the first of many such successes.

B. Applications of K-theory

1. *Vector fields on spheres* (1961) [**23**]

Having so spectacularly solved the Hopf invariant one problem, Adams turned next to the vector fields problem. It was natural for him to try cohomology operations here too. A 1960 note [**20**] gave a partial result, and he was still working in cohomology in July, 1961, when he gave a series of lectures in Berkeley. When the solution came, however, it used K-theory and Adams wrote of his cohomological efforts: "The author's work on this topic may be left in decent obscurity, like the bottom nine-tenths of an iceberg." Write $n = (2a+1)2^b$ and $b = c + 4d$ and let $\rho(n) = 2^c + 8d$. Hurwicz-Radon and Eckman had shown that there exist $\rho(n) - 1$ linearly independent vector fields on S^{n-1}. Adams proved that there do not exist $\rho(n)$ such fields. It suffices to show that the truncated projective space $\mathbf{R}P^{m+\rho(m)}/\mathbf{R}P^{m-1}$ is not coreducible (the bottom cell is not a retract up to homotopy) for any m. He introduced what are now called the Adams operation ψ^k into real and complex K-theory, he calculated the K-theory of truncated projective spaces, with their Adams operations, and he showed that there is no splitting of their real K-theory which is compatible with the operations. All of Adams' papers are well written, but the exposition in this classic paper is especially lovely.

For background, James, in part, and Atiyah had shown that the bundle $O(n)/O(n-k) \to S^{n-1}$ admits a cross-section if and only if n is a multiple of the order of the image of the canonical line bundle in $\tilde{J}(\mathbf{R}P^{k-1})$, and analogously in the complex and quaternionic cases. Curiously, it was left to Atiyah and Bott to observe that Adams' calculations actually imply that $\widetilde{KO}(\mathbf{R}P^k) \cong \tilde{J}(\mathbf{R}P^k)$. This group is cyclic of order $2^{\varphi(k)}$, where $\varphi(k)$ is the number of j such that $0 < j \leq k$ and $j \equiv 0, 1, 2$, or $4 \bmod 8$. Mark Mahowald discussed the significance of this calculation in his talk.

2. *On complex Stiefel manifolds* (1964, with Grant Walker) [**29**]

It is shown that $U(n)/U(n-k) \to S^{2n-1}$ admits a cross-section if and only if M_k divides n; here $\nu_p(M_k) = \sup\{r+\nu_p(r) \,|\, 1 \le r \le (k-1)/(p-1)\}$ if $p \le k$ and $\nu_p(M_k) = 0$ if $p > k$. Atiyah and Todd had shown that the condition is necessary, and they had conjectured that it is sufficient. As already noted, Atiyah had reduced the problem to a calculation in $\tilde{J}(\mathbb{C}P^{k-1})$, and this paper analyzes $\tilde{J}(\mathbb{C}P^n)$ by the methods of $J(X)$-I,II. It gives a worked example of the general study in those papers.

3. *On the groups* $J(X)$*-I* (1963), *II* (1963), *III* (1963), *IV* (1965) ([**25**], [**28**], [**31**], [**35**])

The program in this fundamentally important cycle of papers is to give effective means for computing the group $J(X) = \tilde{J}(X) \oplus Z$ of fiber homotopy equivalence classes of stable vector bundles over a finite CW-complex X. The basic idea is to give computable upper and lower bounds $J''(X)$ and $J'(X)$ for $J(X)$ and to show that the two bounds coincide. Thus $J(X)$ would be captured in the diagram of epimorphisms

$$\begin{array}{ccc} & \nearrow & J''(X) \\ & & \downarrow \\ KO(X) & \longrightarrow & J(X) \\ & & \downarrow \\ & \searrow & J'(X). \end{array}$$

That $J''(X)$ really is an upper bound depends on the celebrated Adams conjecture: "If k is an integer, X is a finite CW-complex and $y \in KO(X)$, then there exists a non-negative integer $e = e(k, y)$ such that $k^e(\psi^k - 1)y$ maps to zero in $J(X)$."

As Michael Crabb explained in his talk, it is now possible to give a fairly elementary proof of the Adams conjecture. It is fortunate that such an argument was not discovered early on. The proofs of the Adams conjecture by Sullivan and Quillen led to a veritable cornucopia of new mathematics, including localizations and completions of spaces and the higher algebraic K-groups of rings.

J(X)-I. The Adams conjecture is proven if y is a linear combination of $O(1)$ and $O(2)$ bundles or if $X = S^{2n}$ and y is a complex bundle. The proof is based on the Dold theorem mod k: if there is a fiberwise map $E_\xi \to E_\eta$ of degree $\pm k$ on each fiber, then $k^e\xi$ and $k^e\eta$ are fiber homotopy equivalent for some $e > 0$.

J(X)-II. The group $J''(X)$ is specified as $KO(X)/W(X)$, where $W(X)$ is the subgroup generated by all elements $k^{e(k)}(\psi^k - 1)y$ for a suitable function e (independent of y). The cannibalistic classes ρ^k are defined by the formula $\rho^k(\xi) = \varphi^{-1}\psi^k\varphi(1)$ on Spin(8n)-bundles ξ, and it is shown that they can be defined more generally after localization. If ξ and η are fiber homotopy equivalent, then $\rho^k(\xi) = \rho^k(\eta)[\psi^k(1+y)/(1+y)]$ for some $y \in \widetilde{K}(X)$ (independent of k). $J'(X)$ is specified as $KO(X)/V(X)$, where $V(X)$ is the subgroup of those x such that $\rho^k(x) = \psi^k(1+y)/(1+y)$ in $KO(X) \otimes Z[1/k]$ for all $k \neq 0$ and some $y \in \widetilde{KO}(X)$. Explicit computations give the groups $KO(\mathbf{R}P^n) = J''(\mathbf{R}P^n) = J'(\mathbf{R}P^n)$ and $J''(S^n) = J'(S^n)$. The latter calculations imply that $J(\pi_{8n+i}(SO)) = Z_2$ if $i = 0$ or 1 and that $J(\pi_{4n-1}(SO))$ is cyclic of order $m(2n)$, where $m(2n)$ is the denominator of $B_n/4n$, although Adams was left with an ambiguity when n is even because he only had the complex and not the real Adams conjecture for bundles over spheres. Of course, these basic calculations are essential to the understanding of the stable homotopy groups of spheres.

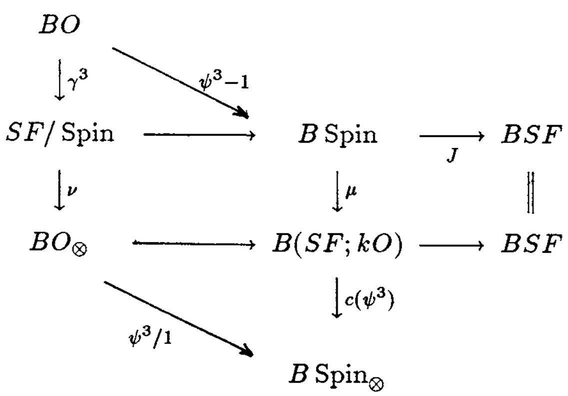
J(X)-III. The main theorem of the series is proven: $J'(X) = J''(X)$. This is based on the fundamental commutative diagram

$$\begin{array}{ccc} \sum_k \widetilde{K}SO(X) & \xrightarrow{\sum k^{e(k)}(\psi^k - 1)} & \widetilde{K}SO(X) \\ \downarrow \sum \vartheta^k & & \downarrow \prod \rho^\ell \\ 1 + \widetilde{K}SO(X) & \xrightarrow{\prod \psi^\ell/1} & \prod_\ell 1 + \widetilde{K}SO(X) \otimes Z[1/\ell] \end{array}$$

The diagram is obtained by summing individual diagrams for pairs (k, ℓ), and the ϑ^k are constructed in the course of the character theoretic proof. The main theorem follows from the fact that this diagram is a weak pullback. The paper also explains and exploits the modular periodicity of the Adams operations.

The paper has a tantalizing last section. It asks for a theory Sph(X) of stable spherical fibrations in which $J(X)$ is a direct summand mapped to by $KO(X)$; Sph(X) should be represented by $BF \times Z$, where F is the monoid of homotopy equivalences of spheres. It also asks for a theory Sph$(X; kO)$ of kO-oriented stable spherical fibrations and gives a number of probable consequences. With characteristic honesty, Adams wrote of this discussion "I will not call the results "theorems", since the underlying assumptions have not been stated precisely enough." This section makes vividly clear just how prescient this whole series of papers was. Many relevant and now standard tools were unavailable to Adams, but he foresaw much that would later be formulated and proven with them.

For example, an alternative version of the diagram above can be constructed conceptually by exploiting localized classifying spaces rather than representation theory. At $p = 2$, the relevant diagram is:



Here $B(SF; kO)$ classifies $\mathrm{Sph}(F; kO)$, $c(\psi^3)$ is the universal cannibalistic class determined by ψ^3, and μ is given by the Atiyah-Bott-Shapiro orientation. The rows are fibration sequences, so μ determines ν and the Adams conjecture determines γ^3. The composite $c(\psi^3)\circ\mu$ is ρ^3, the composite $\nu\circ\gamma^3$ can be taken as ϑ^3, and these two maps are 2-local equivalences.[4] I discussed this approach with Frank, who had envisioned something of the sort. He very much liked it, but he rightly emphasized that you can't proceed this way before you have the Adams conjecture.

J(X)-IV. The results of I–III are applied to computations in the stable homotopy groups of spheres. The starting point is an abstract analysis of the "d and e invariants" of a half exact functor k from the homotopy category to an abelian category $\mathcal{A}$. For $f : X \to Y$, $d(f) = f^* \in \mathrm{Hom}(k(Y), k(X))$. If $d(f) = 0$ and $d(\Sigma f) = 0$, then $e(f)$ is the class of

$$0 \to k(\Sigma X) \to k(Cf) \to k(Y) \to 0$$

in $\mathrm{Ext}^1(k(Y), k(\Sigma X))$. In the applications, $\mathcal{A}$ consists of finitely generated Abelian groups with Adams operations, k is taken to be $\widetilde{K}$ or $\widetilde{KO}$, and X and Y are taken to be spheres or Moore spaces, for which the Hom and Ext target groups are readily computed.

[4]For details, see Chapter V of [J. P. May (with contributions by Nigel Ray, Frank Quinn, and J. Tornehave) E_∞ ring spaces and E_∞ ring spectra. Springer Lecture Notes in Mathematics Vol 577, 1977].

Calculations of these invariants on the groups π_r^s for $r > 1$ are related to $J : \pi_r(SO) \to \pi_r^s$. Here $d = 0$ except in the real case with $r \equiv 1$ or 2 mod 8, when d detects a direct summand $Z_2 \not\subset \mathbf{Im}\, J$ generated by elements μ_r. For any r, the real e invariant detects $\mathbf{Im}\, J$ as a direct summand, the case $r \equiv 7$ mod 8 being incomplete in the paper since the full Adams conjecture was not yet available. Various composition products and Toda brackets are detected by means of Yoneda and Massey products in the target groups. This seems a little like a magical boot strap operation since the Hom and Ext calculations involved are fairly elementary. The conception is a marvelous example of algebraic modeling of topological phenomena.

The complex e-invariant is determined by the Chern character, and, via Adams' paper on the Chern character, to be discussed shortly, this leads to a proof by K-theory of the Hopf invariant one problem for any prime p. Finally, the e-invariant is used to prove that if Y is the mod p^f Moore space for an odd prime p (with bottom cell in a suitable odd dimension) and if $r = 2(p-1)p^{f-1}$, then there is a map $A : \Sigma^r Y \to Y$ which induces an isomorphism on $\widetilde{K}$, so that all of the iterates of A are essential. You have heard about these vitally important periodicity maps in several talks, for example those of Katsumi Shimomura, Doug Ravenel, and especially that of Pete Bousfield.

4. *K-theory and the Hopf Invariant* (1964, with Michael Atiyah) [**34**]
This paper gives the beautiful and definitive K-theoretic proof of the Hopf invariant one result for all primes p. For $p = 2$, it is based on the relation $\psi^2\psi^3 = \psi^3\psi^2$ applied in the obvious 2-cell complex. Alain Jeanneret showed us that this trick can still be used to good effect to obtain new results.

5. *Geometric dimension of bundles over* $\mathbf{R}P^n$ (1974) [**51**]
Since Kee Y. Lam's talk gave a rather complete summary of the results in this nice paper, I will not discuss its main thrust. However, in view of the current interest in periodicity maps, I want to mention an addendum that it gives to the discussion of Moore spaces in J(X)-IV: for $n \geq 5$, there is a map $\Sigma^8 Y_n \to Y_n$ which induces an isomorphism on $\widetilde{KSp}$, where Y_n is the mod 2 Moore space with bottom cell in dimension n.

C. Characteristic classes and calculations in K-theory and cobordism

1. *On formulae of Thom and Wu* (1961) [**19**]
In this beautiful early example of modern algebraic modeling, Adams shows that any Poincaré duality algebra over the Steenrod algebra has "Wu

classes" and thus Stiefel Whitney classes which satisfy all of the same formulas which relate these classes in the cohomology of differential manifolds. The proof is based on the construction and analysis of a suitable universal left and right A-algebra.

2. *On Chern characters and the structure of the unitary group* (1960) **[18]**

Using Bott periodicity to study the Postnikov system of $BU[2q, \ldots, \infty)$, Adams defines characteristic classes $ch_{q,r}(\xi) \in H^{2q+2r}(X;\mathbf{Z})$ for stable bundles ξ over (2q-1)-connected spaces. If ch_r is the r^{th} component of the Chern character, then $ch_{q,r}(\xi)$ rationalizes to $m(r)ch_{q+r}(\xi)$, and the $ch_{q,r}(\xi)$ relate appropriately to Steenrod operations when reduced mod p. Yuli Rudjak noted that some of the ideas Adams introduced here are relevant to the study of the orientability of various kinds of bundles.

3. *Chern characters revisited* (1971) **[47]**

This gives a more modern and sophisticated approach to the Chern character. The image of $H_*(bu;\mathbf{Z})$ in $H_*(bu;\mathbf{Q}) = \mathbf{Q}[u]$, $\deg u = 2$, is shown to be the subgroup generated by $\{u^r/m(r) \mid r \geq 0\}$. The elegant one prime at a time proof is based on the simple A-module structure of $H^*(bu;\mathbf{Z}_p)$. Viewing ch_r as a map $bu \to K(\mathbf{Q}, 2r)$, it follows that the image of $m(r)ch_r : bu_n(X) \to H_{n-2r}(X;\mathbf{Q})$ is integral for any X.

4. *The Hurewicz homomorphism for MU and BP* (1970, with Arunas Liulevicius) **[45]**

This paper gives a nice proof via the Adams spectral sequence of an interpretation of the Hattori-Stong theorem: $\pi_*(BP) \to \pi_*(k \wedge BP)$ is a split monomorphism, and similarly for MU, where k represents connective K-theory.

This is one of the very few papers in which Adams allowed his coauthor to do the actual writing. Adams preferred to hold pen in hand himself, although he paid careful attention to the suggestions of his collaborators.

The following five papers can be viewed as a series in which Adams applied to K-theory the algebraic foundations that he established for the calculational study of generalized homology and cohomology theories.

5. *Hopf algebras of cooperations for real and complex K-theory* (1970, with Albert Harris and Robert Switzer) **[42]**

Using the stable Adams operations in localized K-theory to obtain integrality conditions, $K_*(K)$ is computed as a subring of $K_*(K) \otimes \mathbf{Q}$, which is a ring of finite Laurent series on two variables; $KO_*(KO)$ is also determined. Francis Clarke showed us how to relate this to the study of the ring of cooperations in elliptic homology.

6. *Operations of the Nth kind in K-theory* (1972) [**48**]

In a report on work with David Baird, Adams indicates that, for K-theory localized at an odd prime p, $\mathrm{Ext}^{s,t}_{K_*(K)}(\widetilde{K}_*(X), \widetilde{K}_*(Y)) = 0$ for all finite X and Y and all t when $s \geq 3$. He then speculates about stable homotopy theory "seen through the spectacles of K-theory". Pete Bousfield's beautiful talk on the structure of stable homotopy theory localized at K showed us how these speculations have come to fruition.

7. *Operations on K-theory of torsion-free spaces* (1975, with Peter Hoffman) [**54**]

For integers $n < m$, this paper computes the ring of those operations $K(X) \to K(X)$ which are defined and natural for CW-spectra X such that $\pi_r(X) = 0$ for $r < 2n$, $H_r(X)$ is free for all r, and $H_r(X) = 0$ for $r > 2m$.

8. *Stable operations on complex K-theory* (1976, with Francis Clarke) [**59**]

It is observed that although linear combinations of ψ^1 and ψ^{-1} are the only obvious stable operations, $K^0(K)$ is actually uncountable.

9. *Primitive elements in the K-theory of BSU* (1975) [**56**]

The kernel and cokernel of $PK^0(BU; k) \to PK^0(BSU; k)$ are computed for any commutative ring k. In particular, somewhat surprisingly, it is found that $PK^0(BU; \widehat{Z}_p) \to PK^0(BSU; \widehat{Z}_p)$ is an isomorphism.

D. Stable homotopy and generalized homology

Especially during his last few years at Manchester, Adams made frequent extended trips to the United States. His usual destination was Chicago, a place where he always felt very comfortable and at home. Some of his most influential writing is in notes prepared for delivery in lecture series at Chicago (1967, 1970, and 1971) and at a conference in Seattle (1968). According to Nigel Ray, he tried out some of these lectures on people at Manchester.

1967 *S. P. Novikov's work on operations on complex cobordism* * [**49**][5]

Novikov's work in question was only available in Russian at the time, and it was quite difficult reading even for those who knew Russian. Adams' clear exposition allowed the quick assimilation of this material into the main stream of algebraic topology in the West.

1968 *Lectures on generalized homology* (Seattle) [**39**]

1. *The universal coefficient theorem and the Künneth theorem*

[5] The three lecture notes denoted * are in the "Chicago blue book" (titled after the 1971 lectures). The University of Chicago Press will keep it in print, and I would like to be told if anybody has trouble obtaining a copy.

This classic account shows that the four "UCT's" imply the four "KT's" by specialization, that two of the UCT's imply the other two by duality, and that one UCT can be viewed as a special case of the ASS (Adams' preferred abbreviation for the Adams spectral sequence). It gives a treatment of the remaining UCT, by Atiyah's method in K-theory, that still seems to represent the state of the art. The account is applied to the Conner-Floyd theorem and to other relations between K and MU.

2. *The Adams spectral sequence*

This account of the generalized ASS shows much progress beyond earlier tries at generalization. The now generally accepted preference for homology over cohomology is expounded. Convergence is not studied here.

3. *Hopf algebra and comodule structure*

Definitive foundations are given for the algebra used to describe E_2 of the generalized ASS in terms of homology. The material here was taken for granted in quite a few talks at this conference, for example those of Doug Ravenel, Katsumi Shimomura, and Vladimir Vershinin.

4. *Splitting generalized cohomology theories with coefficients*

This gives a splitting of KU and a parallel splitting of MU via idempotents; the former is still the standard reference, but the latter was soon superseded by Quillen's approach via formal group laws.

5. *Finiteness theorems*

A systematic generalized treatment of coherent rings is given. One application gives that, for finite CW-complexes X, $MU^*(X)$ admits a finite MU^*-resolution by finitely generated free modules (as was also shown by Conner and Smith). Another application (due to J. Cohen) shows that a space Y with non-trivial reduced mod p cohomology has infinitely many non p-trivial stable homotopy groups.

1970 *Quillen's work on formal groups and complex cobordism* * [**49**]

Just as the 1967 lectures allowed the rapid assimilation of Novikov's work, so these lectures allowed the rapid assimilation of Quillen's work. I remember these lectures as great fun. The first eight strictly alternated algebra and topology, giving a connected development of the theory of formal groups on the one hand and a clear exposition of their role in topology on the other. The calculations of $MU^*(MU)$ and $BP^*(BP)$ of Novikov and Quillen were reworked as explicit calculations of $MU_*(MU)$ and $BP_*(BP)$. These calculations have been cited in several talks here.

An interesting survey, *Algebraic topology in the last decade* [**43**], based on a lecture given at a conference at the University of Wisconsin, also dates