Sporadic Groups is the first step in a program to provide a uniform, self-contained treatment of the foundational material on the sporadic simple groups necessary for the classification of the finite simple groups. The classification of the finite simple groups is one of the premier achievements of modern mathematics. It demonstrates that each finite simple group is either a finite analogue of a simple Lie group or one of twenty-six pathological sporadic groups. *Sporadic Groups* provides for the first time a self-contained treatment of the foundations of the theory of sporadic groups accessible to mathematicians with a basic background in finite groups such as in the author's text *Finite Group Theory*.

Introductory material useful for studying the sporadics, such as a discussion of large, extraspecial 2-subgroups and Tits's coset geometries, opens the book. A construction of the Mathieu groups as the automorphism groups of Steiner systems follows. The Golay and Todd modules and the 2-local geometry for M_{24} are discussed. This is followed by the standard construction of Conway of the Leech lattice and the Conway group. The Monster is constructed as the automorphism group of the Griess algebra using some of the best features of the approaches of Griess, Conway, and Tits, plus a few new wrinkles. The existence treatment finishes with an application of the theory of large extraspecial subgroups to produce the twenty sporadics involved in the Monster.

The Aschbacher–Segev approach addresses the uniqueness of the sporadics via coverings of graphs and simplicial complexes. The basics of this approach are developed and used to establish the uniqueness of five of the sporadics.

Researchers in finite group theory will find this text invaluable. The subjects treated will interest combinatorists, number theorists, and conformal field theorists.

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104 Sporadic groups

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Contents

Preface

The classification of the finite simple groups says that each finite simple group is isomorphic to exactly one of the following:

A group of prime order
An alternating group A_n of degree n
A group of Lie type
One of twenty-six sporadic groups

As a first step in the classification, each of the simple groups must be shown to exist and to be unique subject to suitable hypotheses, and the most basic properties of the group must be established. The existence of the alternating group A_n comes for free, while the representation of A_n on its n-set makes possible a first uniqueness proof and easy proofs of most properties of the group. The situation with the groups of Lie type is more difficult, but while groups of Lie rank 1 and 2 cause some problems, Lie theory provides proofs of the existence, uniqueness, and basic structure of the groups of Lie type in terms of their Lie algebras and buildings.

However, the situation with the sporadic groups is less satisfactory. Much of the existing treatment of the sporadic groups remains unpublished and the mathematics which does appear in print lacks uniformity, is spread over many papers, and often depends upon machine calculation.

Sporadic Groups represents the first step in a program to provide a uniform, self-contained treatment of the foundational material on the sporadic groups. More precisely our eventual aim is to provide complete proofs of the existence and uniqueness of the twenty-six sporadic groups subject to appropriate hypotheses, and to derive the most basic structure of the sporadics, such as the group order and the normalizers of subgroups of prime order.

While much of this program is necessarily technical and specialized, other parts are accessible to mathematicians with only a basic knowledge of finite group theory. Moreover some of the sporadic groups are the automorphism groups of combinatorial objects of independent interest, so it is desirable to make this part of the program available to as large an audience as possible. For example, the Mathieu groups are the automorphism groups of Steiner systems and Golay codes while the largest Conway group is the automorphism group of the Leech lattice.

Sporadic Groups begins the treatment of the foundations of the sporadic groups by concentrating on the most accessible chapters of the subject. It is our hope that large parts of the book can be read by the nonspecialist and provide a good picture of the structure of the sporadics and the methods for studying these groups. At the same time the book provides the basis for a complete treatment of the sporadics.

The book is divided into three parts: Part I, introductory material (Chapters 1–5); Part II, existence theorems (Chapters 6–11); and Part III, uniqueness theorems (Chapters 12–17).

The goal of the existence treatment is to construct the largest sporadic group (the Monster) as the group of automorphisms of the Griess algebra. Twenty of the twenty-six sporadic groups are sections of the Monster. We establish the existence of these groups via these embeddings. To construct the Griess algebra one must first construct the Leech lattice and the Conway groups, and to construct the Leech lattice one must first construct the Mathieu groups, their Steiner systems, and the binary Golay code.

There are many constructions of the Mathieu groups. Our treatment proceeds by constructing the Steiner systems for the Mathieu groups as a tower of extensions of the projective plane of order 4. This method has the advantage of supplying the extremely detailed information about the Mathieu groups, their Steiner systems, and the Golay code module and Todd module necessary both for the construction of the Leech lattice and the Griess algebra, and for the proof of the uniqueness of various sporadics.

The construction given here of the Leech lattice and the subgroups stabilizing various sublattices is the standard one due to Conway in [Co1] and [Co2]. The construction of the Griess algebra combines aspects of the treatments due to Griess [Gr2], Conway [Co3], and Tits [T2], plus a few extra wrinkles. The basis of the construction is Parker's loop and Conway's construction via the Parker loop of the normalizer $\bar{N}$ of a certain 4-subgroup of the Monster. Chapter 4 contains a discussion of a general class of loops which includes the Parker loop. This discussion contains much material not needed to construct the Parker loop or the Griess algebra, but the extra discussion provides a context which hopefully makes the Parker loop and Conway's construction of $\bar{N}$ more natural.

The majority of the sporadic groups contain a *large extraspecial* 2-subgroup. Such subgroups provide one of the unifying features of our treatment. The basic theory of large extraspecial subgroups is developed in Chapter 2. The theory is used to recognize and establish the simplicity of the sporadics contained in the Monster that are not symmetry groups of any nice structure.

The eventual object of the uniqueness treatment is to prove each sporadic is unique subject to suitable hypotheses. Here is a typical hypothesis; let w be a positive integer and L a group. (See Chapter 2 for terminology and notation.)

Hypothesis $\mathcal{H}(w, L)$**:** *G is a finite group containing an involution z such that $F^*(C_G(z)) = Q$ is an extraspecial 2-subgroup of order 2^{2w+1}, $C_G(z)/Q \cong L$, and z is not weakly closed in Q with respect to G.*

For example, Hypothesis $\mathcal{H}(12, Co_1)$ characterizes the Monster. Hypotheses of this sort are the appropriate ones for characterizing the sporadics for purposes of the classification.

Sporadic Groups lays the foundation for a proof of the uniqueness of each of the sporadics and supplies actual uniqueness proofs for five of the sporadic groups: M_{24}, He, J_2, Suz, and Co_1.

Our approach to the uniqueness problem follows Aschbacher and Segev in [AS1]. Namely given a group theoretic hypothesis $\mathcal{H}$ we associate to each group G satisfying $\mathcal{H}$ a coset graph Δ defined by some family $\mathcal{F}$ of subgroups of G. We prove the amalgam $\mathcal{A}$ of $\mathcal{F}$ is determined up to isomorphism by $\mathcal{H}$ independently of G, and form the free amalgamated product $\tilde{G}$ of $\mathcal{A}$ and its coset graph $\tilde{\Delta}$. Now there exists a covering $d : \tilde{\Delta} \to \Delta$ of graphs. To complete the proof we show Δ is simply connected so d is an isomorphism and hence $G = \tilde{G}$ is determined up to isomorphism by $\mathcal{H}$.

After developing the most basic part of the conceptual base for our treatment of the sporadic groups in Part I, Chapter 5 closes the first part of the book with an overview of the sporadic groups including the hypotheses by which we expect each group to be characterized, the approach for constructing each of the twenty sporadics involved in the Monster, and a number of historical remarks.

While *Sporadic Groups* concentrates on some of the most accessible and least technical aspects of the study of the sporadic groups, a complete treatment of even this material sometimes requires some difficult specialized arguments. The reader wishing to minimize contact with such arguments can do so as follows. As a general rule the book becomes progressively more difficult in the later chapters. Thus most of the material in Part I should cause little difficulty. A possible exception is Chapter 4, containing the discussion of loops. However, much of this material is not needed in the rest of the book, and none is needed outside of Chapter 10, where the Griess algebra is constructed. As Chapter 10 is the most technical part of Part II, some readers may wish to skip both Chapter 4 and Chapter 10.

Part II contains constructions of the Mathieu groups, the Conway group Co_1 and its sporadic sections, and the Monster and its sporadic sections. Two chapters are devoted to the Mathieu groups and two to the Conway groups. In each case the second of the two chapters is the most technical. Thus the reader may wish to read Chapters 6 and 8, while skipping or skimming Chapters 7 and 9. As suggested in the previous paragraph, dilettantes should skip the construction in Chapter 10 of the Griess algebra and the Monster. The existence proofs for the sporadic sections of the Monster not contained in Co_1 appear in the very short Chapter 11.

The Steiner systems and Golay codes associated to the Mathieu groups and the Leech lattice associated to the Conway groups are beautiful and natural objects. Most of the discussion of these objects appears in Chapters 6 and 8. There is some evidence that the Griess algebra is also natural, in that it is the 0-graded submodule of a conformal field theory preserved by the Monster (cf. [FLM]). However, the construction of the Griess algebra in Chapter 10 is not particularly natural or edifying.

The first two chapters of Part III provide the conceptual base for proving the uniqueness of the sporadic groups. These chapters are fairly elementary. Sections 39 through 41 establishing the uniqueness of M_{24} and $L_5(2)$ probably provide the easiest example of how to apply this machinery to establish uniqueness. On the other hand the proofs of the uniqueness of He, J_2, Suz, and Co_1, while more difficult, are also more representative of the complexity involved in proving the uniqueness of the sporadic groups.

The book closes with tables describing the basic structure of the five sporadic groups considered in detail in *Sporadic Groups*: M_{24}, He, J_2, Suz, and Co_1. These tables enumerate the subgroups of prime order of each group G and the normalizers of these subgroups. Much of this information comes out during the proof of the uniqueness of G, but some of the loose ends are tied up in Chapter 17.