PART I

Chapter 1

Preliminary Results

We take as our starting point the text *Finite Group Theory* [FGT], although we need only a fraction of the material in that text. Frequently quoted results from [FGT] will be recorded in this chapter and in other of the introductory chapters.

Chapters 1 and 2 record some of the most basic terminology and notation we will be using plus some elementary results. The reader should consult [FGT] for other basic group theoretic terminology and notation, although we will try to recall such notation when it is first used, or at least give a specific reference to [FGT] at that point. There is a "List of Symbols" at the end of [FGT] which can be used to help hunt down notation.

We begin in Section 1 with a brief discussion of abstract representations of groups. Then in Section 2 we specialize to permutation representations. In Section 3 we consider graphs and in Section 4 geometries (in the sense of J. Tits) and geometric complexes. In the last few sections of the chapter we record a few basic facts about the general linear group and fiber products of groups.

1. Abstract representations

Let $\mathcal{C}$ be a category. For X an object in $\mathcal{C}$, we write $Aut(X)$ for the group of automorphisms of X under the operation of composition in $\mathcal{C}$ (cf. Section 2 in [FGT]). A *representation* of a group G in the category $\mathcal{C}$ is a group homomorphism $\pi; G \to Aut(X)$. For example, a *permutation representation* is a representation in the category of sets and a *linear* *representation* is a representation in the category of vector spaces and linear maps.

If $\alpha : A \to B$ is an isomorphism of objects in $\mathcal{C}$ then α induces a map

$$\alpha^* : Mor(A, A) \to Mor(B, B)$$
$$\beta \mapsto \alpha^{-1}\beta\alpha$$

and α^* restricts to an isomorphism $\alpha^* : Aut(A) \to Aut(B)$. Thus in particular if $A \cong B$ then $Aut(A) \cong Aut(B)$.

A representation $\pi : G \to Aut(A)$ is *faithful* if π is injective.

Two representations $\pi : G \to Aut(A)$ and $\sigma : G \to Aut(B)$ in $\mathcal{C}$ are *equivalent* if there exists an isomorphism $\alpha : A \to B$ such that $\sigma = \pi\alpha^*$ is the composition of π with α^*. Equivalently for all $g \in G$, $(g\pi)\alpha = \alpha(g\sigma)$.

Similarly if $\pi_i : G_i \to Aut(A_i)$, $i = 1, 2$, are representations of groups G_i on objects A_i in $\mathcal{C}$, then π_1 is said to be *quasiequivalent* to π_2 if there exists a group isomorphism $\beta : G_1 \to G_2$ and an isomorphism $\alpha : A_1 \to A_2$ such that $\pi_2 = \beta^{-1}\pi_1\alpha^*$. Observe that we have a permutation representation of $Aut(G)$ on the equivalence classes of representations of G via $\alpha : \pi \mapsto \alpha\pi$ with the orbits the quasiequivalence classes. Write $Aut(G)_\pi$ for the stabilizer of the equivalence class of π under this representation. The following result is Exercise 1.7 in [FGT]:

Lemma 1.1: *Let $\pi, \sigma : G \to Aut(A)$ be faithful representations. Then*

(1) π is quasiequivalent to σ if and only if $G\pi$ is conjugate to $G\sigma$ in $Aut(A)$.

(2) $Aut_{Aut(A)}(G\pi) \cong Aut(G)_\pi$.

If $H \leq G$ then write $Aut_G(H) = N_G(H)/C_G(H)$ for the group of automorphisms of H induced by G. Also

$$C_G(H) = \{c \in G : ch = hc \text{ for all } h \in H\}$$

is the *centralizer* in G of H and $N_G(H)$ is the *normalizer* in G of H, that is, the largest subgroup of G in which H is normal.

2. Permutation representations

In this section X is a set. We refer the reader to Section 5 of [FGT] for our notational conventions involving permutation groups, although we record a few of the most frequently used conventions here. In particular we write $Sym(X)$ for the symmetric group on X and if X is finite we write $Alt(X)$ for the alternating group on X. Further S_n, A_n denote the symmetric and alternating groups of degree n; that is, $S_n = Sym(X)$ and $A_n = Alt(X)$ for X of order n.

Let $\pi : G \to Sym(X)$ be a permutation representation of a group G on X. Usually we suppress π and write xg for the image $x(g\pi)$ of a point $x \in X$ under the permutation $g\pi$, $g \in G$. For $S \subseteq G$, we write $Fix(S) = Fix_X(S)$ for the set of fixed points of S on X. For $Y \subseteq X$,

$$G_Y = \{g \in G : yg = y \text{ for all } y \in Y\}$$

is the *pointwise stabilizer* of Y in G,

$$G(Y) = \{g \in G : Yg = Y\}$$

is the *global stabilizer* of Y in G, and $G^Y = G(Y)/G_Y$ is the image of $G(Y)$ in $Sym(Y)$ under the restriction map. In particular G_y denotes the stabilizer of a point $y \in X$.

Recall the *orbit* of $x \in X$ under G is $xG = \{xg : g \in G\}$ and G is *transitive* on X if G has just one orbit on X. If G is transitive on X then our representation π is equivalent to the representation of G by right multiplication on the coset space G/G_x via the map $G_xg \mapsto xg$ (cf. 5.9 in [FGT]).

A subgroup K of G is a *regular normal subgroup* of G if $K \trianglelefteq G$ and K is *regular* on X; that is, K is transitive on X and $K_x = 1$ for $x \in X$.

Recall a transitive permutation group G is *primitive* on X if G preserves no nontrivial partition on X. Further G is primitive on X if and only if G_x is maximal in G (cf. 5.19 in [FGT]).

Lemma 2.1: *Let G be transitive on X, $x \in X$, and $K \leq G$. Then*

(1) K is transitive on X if and only if $G = G_xK$.
(2) If $1 \neq K \trianglelefteq G$ and G is primitive on X then K is transitive on X.
(3) If K is a regular normal subgroup of G then the representations of G_x on X and on K by conjugation are equivalent.

Proof: These are all well known; see, for example, 5.20, 15.15, and 15.11 in [FGT].

Recall that G is *t-transitive* on X if G is transitive on ordered t-tuples of distinct points of X. In Chapter 6 we will find that the Mathieu group M_{m+t} is t-transitive on $m+t$ points for $m = 19$ and $t = 3, 4, 5$ and $m = 7$ and $t = 4, 5$.

Lemma 2.2: *Let G be t-transitive on a finite set X with $t \geq 2$, $x \in X$, and $1 \neq K \trianglelefteq G$. Then*

(1) G is primitive on X.
(2) K is transitive on X and $G = G_xK$.

(3) *If K is regular on X then $|K| = |X| = p^e$ is a power of some prime p, and if $t > 2$ then $p = 2$.*
(4) *If $t = 3 < |X|$ and $|G : K| = 2$ then K is 2-transitive on X.*

Proof: Again these are well-known facts. See, for example, 15.14 and 15.13 in [FGT] for (1) and (3), respectively. Part (2) follows from (1) and 1.1. Part (4) is left as Exercise 1.1.

3. Graphs

A *graph* $\Delta = (\Delta, *)$ consists of a set Δ of *vertices* (or objects or points) together with a symmetric relation $*$ called *adjacency* (or incidence or something else). The ordered pairs in the relation are called the *edges* of the graph. We write $u * v$ to indicate two vertices are related via $*$ and say u is *adjacent* to v. Denote by $\Delta(u)$ the set of vertices adjacent to u and distinct from u and define $u^\perp = \Delta(u) \cup \{u\}$.

A *path of length n* from u to v is a sequence of vertices $u = u_0, u_1, \ldots, u_n = v$ such that $u_{i+1} \in u_i^\perp$ for each i. Denote by $d(u, v)$ the minimal length of a path from u to v. If no such path exists set $d(u, v) = \infty$. $d(u, v)$ is the *distance* from u to v.

The relation $\sim$ on Δ defined by $u \sim v$ if and only if $d(u, v) < \infty$ is an equivalence relation on Δ. The equivalence classes of this relation are called the *connected components* of the graph. The graph is *connected* if it has just one connected component. Equivalently there is a path between any pair of vertices.

A *morphism* of graphs is a function $\alpha : \Delta \to \Delta'$ from the vertex set of Δ to the vertex set of Δ' which preserves adjacency; that is, $u^\perp\alpha \subseteq (u\alpha)^\perp$ for each $u \in \Delta$.

A group G of automorphisms of Δ is *edge transitive* on Δ if G is transitive on Δ and on the edges of Δ.

Representations of groups on graphs play a big role in this book. For example, we prove the uniqueness of some of the sporadics G by considering a representation of G on a suitable graph. The following construction supplies us with such graphs.

Let G be a transitive permutation group on a finite set Δ. Recall the *orbitals* of G on Δ are the orbits of G on the set product $\Delta^2 = \Delta \times \Delta$. The *permutation rank* of G is the number of orbitals of G; recall this is also the number of orbits of G_x on Δ for $x \in \Delta$.

Given an orbital Ω of G, the *paired orbital* Ω^p of Ω is

$$\Omega^p = \{(y, x) : (x, y) \in \Omega\}.$$

Evidently Ω^p is an orbital of G with $(\Omega^p)^p = \Omega$. The orbital Ω is said to be *self-paired* if $\Omega^p = \Omega$. For example, the *diagonal orbital* $\{(x,x) : x \in \Delta\}$ is a self-paired orbital.

Lemma 3.1: *(1) A nondiagonal orbital $(x,y)G$ of G is self-paired if and only if (x,y) is a cycle in some $g \in G$.*

(2) If G is finite then G possesses a nondiagonal self-paired orbital if and only if G is of even order.

(3) If G is of even order and permutation rank 3 then all orbitals of G are self-paired.

Proof: See 16.1 in [FGT].

Lemma 3.2: *(1) Let Ω be a self-paired orbital of G. Then Ω is a symmetric relation on Δ, so $\Delta = (\Delta, \Omega)$ is a graph and G is an edge transitive group of automorphisms of Δ.*

(2) Conversely if H is an edge transitive group of automorphisms of a graph $\Delta = (\Delta, *)$ then the set $*$ of edges of Δ is a self-paired orbital of G on Δ, and Δ is the graph determined by this orbital.

Many of the sporadics have representations as rank 3 permutation groups. Indeed some were discovered via such representations; see Chapter 5 for a discussion of the sporadics discovered this way. See also Exercise 16.5, which considers the rank 3 representation of J_2, and Lemmas 24.6, 24.7, and 24.11, which establish the existence of rank 3 representations of Mc, $U_4(3)$, and HS.

In the remainder of this section assume G is of even order and permutation rank 3 on a set X. Hence G has two nondiagonal orbitals Δ and Γ and by 3.1, each is self-paired. Further for $x \in X$, G_x has two orbits $\Delta(x)$ and $\Gamma(x)$ on $X - \{x\}$, where $\Delta(x) = \{y \in X : (x,y) \in \Delta\}$ and $\Gamma(x) = \{z \in X : (x,z) \in \Gamma\}$. By 3.2, $X = (X, \Delta)$ is a graph and G is an edge transitive group of automorphisms of X. Notice $\Delta(x) = X(x)$ in our old notation.

The following notation is standard for rank 3 groups and their graphs: $k = |\Delta(x)|$, $l = |\Gamma(x)|$, $\lambda = |\Delta(x) \cap \Delta(y)|$ for $y \in \Delta(x)$, and $\mu = |\Delta(x) \cap \Delta(z)|$ for $z \in \Gamma(x)$. The integers k, l, λ, μ are the *parameters* of the rank 3 group G. Also let $n = |X|$ be the degree of the representation.

Lemma 3.3: *Let G be a rank 3 permutation group of even order on a finite set of order n with parameters k, l, λ, μ. Then*

(1) $n = k + l + 1$.
(2) $\mu l = k(k - \lambda - 1)$.

(3) If $\mu \neq 0$ or k then G is primitive and the graph $\mathcal{G}$ of G is connected.

(4) Assume G is primitive. Then either

(a) $k = l$ and $\mu = \lambda + 1 = k/2$, or

(b) $d = (\lambda - \mu)^2 + 4(k - \mu)$ is a square and setting $D = 2k + (\lambda - \mu)(k + l)$, $d^{1/2}$ divides D and $2d^{1/2}$ divides D if and only if n is odd.

Proof: See Section 16 of [FGT].

4. Geometries and complexes

In this book we adopt a notion of geometry due to J. Tits in [T1].

Let I be a finite set. For $J \subseteq I$, let $J' = I - J$ be the complement of J in I. A *geometry* over I is a triple $(\Gamma, \tau, *)$ where Γ is a set of objects, $\tau : \Gamma \to I$ is a surjective type function, and $*$ is a symmetric incidence relation on Γ such that objects u and v of the same type are incident if and only if $u = v$. We call $\tau(u)$ the *type* of the object u. Notice $(\Gamma, *)$ is a graph. We usually write Γ for the geometry $(\Gamma, \tau, *)$ and Γ_i for the set of objects of Γ of type i.

The *rank* of the geometry Γ is the cardinality of I.

A *flag* of Γ is a subset T of Γ such that each pair of objects in T is incident. Notice our one axiom insures that if T is a flag then the type function $\tau : T \to I$ is injective. Define the *type* of T to be $\tau(T)$ and the *rank* of T to be the cardinality of T. The *chambers* of Γ are the flags of type I.

A *morphism* $\alpha : \Gamma \to \Gamma'$ of geometries is a function $\alpha : \Gamma \to \Gamma'$ of the associated object sets which preserves type and incidence; that is, if $u, v \in \Gamma$ with $u * v$ then $\tau(u) = \tau'(u\alpha)$ and $u\alpha *' v\alpha$. A group G of automorphisms of Γ is *edge transitive* if G is transitive on flags of type J for each subset J of I of order at most 2. Similarly G is *flag transitive* on Γ if G is transitive on flags of type J for all $J \subseteq I$.

Representations of groups on geometries also play an important role in *Sporadic Groups*. For example, the Steiner systems in Chapter 6 are rank 2 geometries whose automorphism groups are the Mathieu groups. Here are some other examples:

Examples (1) Let V be an n-dimensional vector space over a field F. We associate a geometry $PG(V)$ to V called the *projective geometry* of V. The objects of $PG(V)$ are the proper nonzero subspaces of V, with incidence defined by inclusion. The type of U is $\tau(U) = dim(U)$. Thus

$PG(V)$ is of rank $n-1$. The projective general linear group on V is a flag transitive group of automorphism of $PG(V)$.

(2) A *projective plane* is a rank 2 geometry Γ whose two types of objects are called points and lines and such that:

(PP1) Each pair of distinct points is incident with a unique line.
(PP2) Each pair of distinct lines is incident with a unique point.
(PP3) There exist four points no three of which are on a common line.

Remarks. (1) Rank 2 projective geometries are projective planes.

(2) If Γ is a finite projective plane then there exists an integer q such that each point is incident with exactly $q+1$ lines, each line is incident with exactly $q+1$ points, and Γ has q^2+q+1 points and lines.

Examples (3) If f is a sesquilinear or quadratic form on V then the *totally singular subspaces* of V are the subspaces U such that f is trivial on U. The set of such subspaces forms a subgeometry of the projective geometry. See, for example, page 99 in [FGT].

(4) Let G be a group and $\mathcal{F} = (G_i : i \in I)$ a family of subgroups of G. Define $\Gamma(G, \mathcal{F})$ to be the geometry whose set of objects of type i is the coset space G/G_i and with objects $G_i x$ and $G_j y$ incident if $G_i x \cap G_j y \neq \varnothing$. Observe:

Lemma 4.1: *(1) G is represented as an edge transitive group of automorphisms of $\Gamma(G, \mathcal{F})$ via right multiplication and $\Gamma(G, \mathcal{F})$ possesses a chamber.*

(2) Conversely if H is an edge transitive group of automorphisms of a geometry Γ and Γ possesses a chamber C, then $\Gamma \cong \Gamma(H, \mathcal{F})$, where $\mathcal{F} = (H_c : c \in C)$.

The construction of 4.1 allows us to represent each group G on various geometries. The construction is used in Chapter 13 as part of our machine for establishing the uniqueness of groups. Further the construction associates to each sporadic group G various geometries which can be used to study the subgroup structure of G. The latter point of view is not explored to any extent in *Sporadic Groups*; see instead [A2] or [RS] where such geometries are discussed. We do use the 2-local geometry of M_{24} to study that group in Chapter 7.

Define the *direct sum* of geometries Γ_i on I_i, $i=1,2$, to be the geometry $\Gamma_1 \oplus \Gamma_2$ over the disjoint union I of I_1 and I_2 whose object set is the disjoint union of Γ_1 and Γ_2, whose type function is $\tau_1 \cup \tau_2$, and whose incidence is inherited from Γ_1 and Γ_2 with each object in Γ_1 incident with each object in Γ_2.

Example (5) A *generalized digon* is a rank 2 geometry which is the direct sum of rank 1 geometries. That is, each element of type 1 is incident with each element of type 2.

Lemma 4.2: *Let G be a group and $\mathcal{F} = \{G_1, G_2\}$ a pair of subgroups of G. Then $\Gamma(G, \mathcal{F})$ is a generalized digon if and only if $G = G_1G_2$.*

Proof: As G is edge transitive on Γ, Γ is a generalized digon if and only if G_2 is transitive on Γ_1 if and only if $G = G_1G_2$ by 2.1.1.

Given a flag T, let $\Gamma(T)$ consist of all $v \in \Gamma - T$ such that $v * t$ for all $t \in T$. We regard $\Gamma(T)$ as a geometry over $I - \tau(T)$. The geometry $\Gamma(T)$ is called the *residue* of T.

Example (6) Let $\Gamma = PG(V)$ be the projective geometry of an n-dimensional vector space. Then for $U \in \Gamma$, the residue $\Gamma(U)$ of the object U is isomorphic to $PG(U) \oplus PG(V/U)$.

The category of geometries is not large enough; we must also consider either the category of chamber systems or the category of geometric complexes.

A *chamber system* over I is a set X together with a collection of equivalence relations $\sim_i$, $i \in I$. For $J \subseteq I$ and $x \in X$, let $\sim_J$ be the equivalence relation generated by the relations $\sim_j$, $j \in J$, and $[x]_J$ the equivalence class of $\sim_J$ containing x. Define X to be *nondegenerate* if for each $x \in X$, and $j \in I$, $\{x\} = \bigcap_i [x]_{i'}$ and $[x]_j = \bigcap_{i \in j'} [x]_{i'}$. A morphism of chamber systems over I is a map preserving each equivalence relation.

The notion of "chamber system" was introduced by J. Tits in [T1].

Recall that a *simplicial complex* K consists of a set X of *vertices* together with a distinguished set of nonempty subsets of X called the *simplices* of K such that each nonempty subset of simplex is a simplex. The morphisms of simplicial complexes are the *simplicial maps*; that is, a simplicial map $f : K \to K'$ is a map $f : X \to X'$ of vertices such that $f(s)$ is a simplex of K' for each simplex s of K.

Example (7) If Δ is a graph then the *clique complex* $K(\Delta)$ is the simplicial complex whose vertices are the vertices of Δ and whose simplices are the finite cliques of Δ. Recall a *clique* of Δ is a set Y of vertices such that $y \in x^\perp$ for each $x, y \in Y$. Conversely if K is a simplicial complex then the *graph* of K is the graph $\Delta = \Delta(K)$ whose vertices are the vertices of K and with $x * y$ if $\{x, y\}$ is a simplex of K. Observe K is a subcomplex of $K(\Delta(K))$.

Given a simplicial complex K and a simplex s of K, define the *star* of s to be the subcomplex $st_K(s)$ consisting of the simplices t of K such that

$s \cup t$ is a simplex of K. Define the *link* $Link_K(s)$ to be the subcomplex of $st_K(s)$ consisting of the simplices t of $st_K(s)$ such that $t \cap s = \varnothing$.

A *geometric complex* over I is a geometry Γ over I together with a collection $\mathcal{C}$ of distinguished chambers of Γ such that each flag of rank 1 or 2 is contained in a member of $\mathcal{C}$. The *simplices* of the complex are the subflags of members of $\mathcal{C}$. A morphism $\alpha : \mathcal{C} \to \mathcal{C}'$ of complexes over I is a morphism of geometries with $\mathcal{C}\alpha \subseteq \mathcal{C}'$. Notice a geometric complex is just a simplicial complex together with a type function on vertices that is injective on simplices.

Example (8) The *flag complex* of a geometry Γ is the simplicial complex on Γ in which all chambers are distinguished. Notice the flag complex is a geometric complex if and only if each flag of rank at most 2 is contained in a chamber. Further as a simplicial complex, the flag complex is just the clique complex of Γ regarded as a graph.

Many theorems about geometries are best established in the larger categories of geometric complexes or chamber systems. Theorem 4.11 is an example of such a result. We find in a moment in Lemma 4.3 below that the category of nondegenerate chamber systems is isomorphic to the category of geometric complexes. I find the latter category more intuitive and so work with complexes rather than chamber systems. But others prefer chamber systems and there is a growing literature on the subject.

Given a chamber system X define Γ_X to be the geometry whose objects of type i are the equivalence classes of the relation $\sim_{i'}$ with $A * B$ if and only if $A \cap B \neq \varnothing$. For $x \in X$ let C_x be the set of equivalence classes containing x; thus C_x is a chamber in Γ_X. Define $\mathcal{C}_X$ to be the set of chambers C_x, $x \in X$, of Γ_X. If $\alpha : X \to X'$ is a morphism of chamber systems define $\alpha_{\mathcal{C}} : \mathcal{C}_X \to \mathcal{C}_{X'}$ to be the morphism of complexes such that $\alpha_{\mathcal{C}} : A \mapsto A'$ for A a $\sim_{i'}$ equivalence class of X and A' the $\sim_{i'}$ equivalence class containing $A\alpha$.

Conversely given a geometric complex $\mathcal{C}$ over I let $\sim_i$ be the equivalence relation on $\mathcal{C}$ defined by $A \sim_i B$ if A and B have the same subflag of type i'. Then we have a chamber system $X_{\mathcal{C}}$ with chamber set $\mathcal{C}$ and equivalence relations $\sim_i$. Further if $\alpha : \mathcal{C} \to \mathcal{C}'$ is a morphism of complexes let $\alpha_X : X_{\mathcal{C}} \to X_{\mathcal{C}'}$ be the morphism of chamber systems defined by the induced map on chambers.

Lemma 4.3: *The category of nondegenerate chamber systems over I is isomorphic to the category of geometric complexes over I via the maps $X \mapsto \mathcal{C}_X$ and $\mathcal{C} \mapsto X_{\mathcal{C}}$.*

Example (9) Let G be a group and $\mathcal{F} = (G_i : i \in I)$ a family of subgroups of I. For $J \subseteq I$ and $x \in G$ define

$$S_{J,x} = \{G_j x : j \in J\}.$$

Thus $S_{J,x}$ is a flag of the geometry $\Gamma(G, \mathcal{F})$ of type J. Observe that the stabilizer of the flag $S_J = S_{J,1}$ is the subgroup $G_J = \bigcap_{j \in J} G_j$. Define $\mathcal{C}(G, \mathcal{F})$ to be the geometric complex over I with geometry $\Gamma(G, \mathcal{F})$ and distinguished chambers $S_{I,x}$, $x \in G$. Then $\mathcal{C}(G, \mathcal{F})$ is a geometric complex with simplices $S_{J,x}$, $J \subseteq I$, $x \in G$, and G acts as an edge transitive group of automorphisms of $\mathcal{C}(G, \mathcal{F})$ via right multiplication, and transitively on $\mathcal{C}(G, \mathcal{F})$. Indeed:

Lemma 4.4: *Assume $\mathcal{C}$ is a geometric complex over I and G is an edge transitive group of automorphisms with $\mathcal{C} = CG$ for some $C \in \mathcal{C}$. Let $G_i = G_{x_i}$, where $x_i \in C$ is of type i, and let $\mathcal{F} = (G_i : i \in I)$. Then the map $x_i g \mapsto G_i g$ is an isomorphism of $\mathcal{C}$ with $\mathcal{C}(G, \mathcal{F})$.*

Further we have a chamber system $X(G, \mathcal{F})$ whose chamber set is G/G_I and with $G_I x \sim_i G_I y$ if and only if $xy^{-1} \in G_{i'}$. Observe that the map $G_I x \mapsto S_{I,x}$ defines an isomorphism of the chamber systems $X(G, \mathcal{F})$ and $X_{\mathcal{C}(G,\mathcal{F})}$.

The construction of 4.4 allows us to represent a group G on many complexes. We make use of this construction in Chapter 13 as part of our uniqueness machine.

Let $\mathcal{C} = (\Gamma, \mathcal{C})$ be a geometric complex over I. Given a simplex S of type J, regard the link $Link_{\mathcal{C}}(S)$ of S to be a geometric complex over J'; thus the objects of $Link_{\mathcal{C}}(S)$ of type $i \in J'$ are those $v \in \Gamma_i$ such that $S \cup \{v\}$ is a simplex and with $v * u$ if $S \cup \{u, v\}$ is a simplex, and the chamber set $\mathcal{C}(S)$ of $Link_{\mathcal{C}}(S)$ consists of the simplices $C - S$ with $S \subseteq C \in \mathcal{C}$. For example, $\mathcal{C} = Link_{\mathcal{C}}(\varnothing)$ is the link of the empty simplex. Notice that if all flags are simplices then the geometry of $Link_{\mathcal{C}}(S)$ is the residue $\Gamma(S)$ of S in the geometry Γ.

We say $\mathcal{C}$ is *residually connected* if the link of each simplex of corank at least two (including $\varnothing$ if $|I| \geq 2$) is connected. A geometry Γ is residually connected if each flag is contained in a chamber and the flag complex of Γ is residually connected.

Lemma 4.5: *Let $\mathcal{F} = (G_i : i \in I)$ be a family of subgroups of G. Then*

(1) $\Gamma(G, \mathcal{F})$ is connected if and only if $G = \langle \mathcal{F} \rangle$.
(2) $Link_{\mathcal{C}}(S_J) \cong \mathcal{C}(G_J, \mathcal{F}_J)$ for each $J \subseteq I$, where

$$\mathcal{F}_J = (G_{J \cup \{i\}} : i \in J').$$