Introduction

An *n-dimensional manifold* M is a paracompact Hausdorff topological space such that each point $x \in M$ has a neighbourhood homeomorphic to the Euclidean n-space $\mathbb{R}^n$. The homology and cohomology of a compact n-dimensional manifold M are related by the Poincaré duality isomorphisms

$$H^{n-*}(M) \cong H_*(M) ,$$

using twisted coefficients in the nonorientable case.

An *n-dimensional Poincaré space* X is a topological space such that $H^{n-*}(X) \cong H_*(X)$ with arbitrary coefficients. A Poincaré space is *finite* if it has the homotopy type of a finite CW complex. A compact n-dimensional manifold M is a finite n-dimensional Poincaré space, as is any space homotopy equivalent to M. However, a finite Poincaré space need not be homotopy equivalent to a compact manifold. The *manifold structure existence problem* is to decide if a finite Poincaré space is homotopy equivalent to a compact manifold.

A homotopy equivalence of compact manifolds need not be homotopic to a homeomorphism. The *manifold structure uniqueness problem* is to decide if a homotopy equivalence of compact manifolds is homotopic to a homeomorphism, or at least h-cobordant to one. The mapping cylinder of a homotopy equivalence of compact manifolds is a finite Poincaré h-cobordism with manifold boundary, which is homotopy equivalent rel ∂ to a compact manifold h-cobordism if and only if the homotopy equivalence is h-cobordant to a homeomorphism. The uniqueness problem is thus a relative version of the existence problem.

The Browder–Novikov–Sullivan–Wall surgery theory provides computable obstructions for deciding the manifold structure existence and uniqueness problems in dimensions ≥ 5. The obstructions use a mixture of the topological K-theory of vector bundles and the algebraic L-theory of quadratic forms. A finite Poincaré space is homotopy equivalent to a compact manifold if and only if the Spivak normal fibration admits a topological bundle reduction such that a corresponding normal map from a manifold to the Poincaré space has zero surgery obstruction. A homotopy equivalence of compact manifolds is h-cobordant to a homeomorphism if and only if it is normal bordant to the identity by a normal bordism with zero rel ∂ surgery obstruction. The theory applies in general only in dimensions ≥ 5 because it relies on the Whitney trick for removing singularities, just like the h- and s-cobordism theorems.

The algebraic theory of surgery of Ranicki [140]–[146] is extended here to a combinatorial treatment of the manifold structures existence and problems, providing an intrinsic characterization of the manifold structures in a homotopy type in terms of algebraic transversality properties on the chain level. The Poincaré duality theorem is shown to have a converse: a homotopy type contains a compact topological manifold if and only if it has sufficient local Poincaré duality. A homotopy equivalence of compact manifolds is homotopic to a homeomorphism if and only if the point inverses are algebraic Poincaré null-cobordant. The bundles and normal maps in the traditional approach are relegated from the statements of the results to the proofs.

An *n-dimensional algebraic Poincaré complex* is a chain complex C with a Poincaré duality chain equivalence $C^{n-*} \simeq C$. Algebraic Poincaré complexes are used here to define the *structure groups* $\mathbb{S}_*(X)$ of a space X. The structure groups are the value groups for the obstructions to the existence and uniqueness problems. The *total surgery obstruction* $s(X) \in \mathbb{S}_n(X)$ of an n-dimensional Poincaré space X is a homotopy invariant such that $s(X) = 0$ if (and for $n \geq 5$ only if) X is homotopy equivalent to a compact n-dimensional manifold. The *structure invariant* $s(f) \in \mathbb{S}_{n+1}(M)$ of a homotopy equivalence $f\colon N \longrightarrow M$ of compact n-dimensional manifolds is a homotopy invariant such that $s(f) = 0$ if (and for $n \geq 5$ only if) f is h-cobordant to a homeomorphism.

Chain homotopy theory can be used to decide if a map of spaces is a homotopy equivalence: by Whitehead's theorem a map of connected CW complexes $f\colon X \longrightarrow Y$ is a homotopy equivalence if and only if f induces an isomorphism of the fundamental groups $f_*\colon \pi_1(X) \longrightarrow \pi_1(Y)$ and a chain equivalence $\tilde{f}\colon C(\widetilde{X}) \longrightarrow C(\widetilde{Y})$ of the cellular $\mathbb{Z}[\pi_1(X)]$-module chain complexes of the universal covers $\widetilde{X}, \widetilde{Y}$ of X, Y. It will be shown here that the cobordism theory of algebraic Poincaré complexes can be similarly used to decide the existence and uniqueness problems in dimensions ≥ 5. A finite Poincaré space X is homotopy equivalent to a compact manifold if and only if the Poincaré duality $\mathbb{Z}[\pi_1(X)]$-module chain equivalence $[X] \cap -\colon C(\widetilde{X})^{n-*} \longrightarrow C(\widetilde{X})$ of the universal cover $\widetilde{X}$ is induced up to algebraic Poincaré cobordism by a Poincaré duality of a local system of $\mathbb{Z}$-module chain complexes. A homotopy equivalence of compact manifolds f is h-cobordant to a homeomorphism if and only if the chain equivalence $\tilde{f}$ is induced up to algebraic Poincaré cobordism by an equivalence of local systems of $\mathbb{Z}$-module chain complexes. Such results are direct descendants of the h- and s-cobordism theorems, which provided necessary and sufficient cobordism-theoretic and Whitehead torsion conditions for compact manifolds of dimension ≥ 5 to be homeomorphic.

Generically, *assembly* is the passage from a local input to a global output. The input is usually topologically invariant and the output is homotopy invariant. This is the case in the original geometric assembly map of Quinn, and the *algebraic L-theory assembly* map defined here.

The passage from the topology of compact manifolds to the homotopy theory of finite Poincaré spaces is the assembly of particular interest here. In general, it is not possible to reverse the assembly process without some extra geometric hypotheses. Manifolds of a certain type are said to be *rigid* if every homotopy equivalence is homotopic to a homeomorphism, that is if the uniqueness problem has a unique affirmative solution. The classification of surfaces and their homotopy equivalences shows that compact 2-dimensional manifolds are rigid. Haken 3-dimensional manifolds are rigid, by the result of Waldhausen. The Mostow rigidity theorem for symmetric spaces and related results in hyperbolic geometry give the classic instances of higher dimensional manifolds with rigidity. The *Borel conjecture* is that every aspherical Poincaré space $B\pi$ is homotopy equivalent to a compact aspherical topological manifold, and that any homotopy equivalence of such manifolds is homotopic to a homeomorphism. Surgery theory has provided many examples of groups π with sufficient geometry to verify both this conjecture and the closely related *Novikov conjecture* on the homotopy invariance of the higher signatures. The rigidity of aspherical manifolds with fundamental group π is equivalent to the algebraic L-theory assembly map for the classifying space $B\pi$ being an isomorphism. The more complicated homotopy theory of manifolds with non-trivial higher homotopy groups is reflected in non-rigidity, with a corresponding deviation from isomorphism in the algebraic L-theory assembly map.

The Leray homology spectral sequence for a map $f:Y\longrightarrow X$ can be viewed as an assembly process, with input the E^2-terms

$$E^2_{p,q} \ = \ H_p(X;\{H_q(f^{-1}(x))\})$$

and output the E^∞-terms associated to $H_*(Y)$. The spectral sequence can be used to prove the Vietoris–Begle mapping theorem: if f is a map between reasonable spaces (such as paracompact polyhedra) with acyclic point inverses $f^{-1}(x)$ $(x\in X)$ then f is a homology equivalence. The topologically invariant local condition of f inducing isomorphisms

$$(f|)_* \ : \ H_*(f^{-1}(x)) \xrightarrow{\simeq} H_*(\{x\}) \ \ (x\in X)$$

assembles to the homotopy invariant global condition of f inducing isomorphisms

$$f_* \ : \ H_*(Y) \xrightarrow{\simeq} H_*(X) \ .$$

There is also a cohomology version, with input

$$E_2^{p,q} \ = \ H^p(X;\{H^q(f^{-1}(x))\})$$

and output $H^*(Y)$. The dihomology spectral sequences of Zeeman [188] can be similarly viewed as assembly processes, piecing together the homology (resp. cohomology) of a space X from the cohomology (resp. homology) with coefficients in the local homology (resp. cohomology). The homology version has input

$$E_2^{p,q} \;=\; H^p(X;\{H_{n-q}(X,X\backslash\{x\})\})$$

and output $H_{n-*}(X)$, for any $n\in\mathbb{Z}$. The cohomology version has input

$$E^2_{p,q} \;=\; H_p(X;\{H^{n-q}(X,X\backslash\{x\})\})$$

and output $H^{n-*}(X)$.

An *n-dimensional homology manifold* X is a topological space such that the local homology groups at each point $x\in X$ are the local homology groups of $\mathbb{R}^n$

$$H_*(X,X\backslash\{x\}) \;=\; H_*(\mathbb{R}^n,\mathbb{R}^n\backslash\{0\}) \;=\; \begin{cases}\mathbb{Z} & \text{if } *=n\\ 0 & \text{if } *\neq n\ .\end{cases}$$

For compact X the local fundamental classes $[X]_x\in H_n(X,X\backslash\{x\})$ assemble to a global fundamental class $[X]\in H_n(X)$, using twisted coefficients in the nonorientable case. The dihomology spectral sequences collapse for a compact homology manifold X, assembling the local Poincaré duality isomorphisms

$$[X]_x\cap - \;:\; H^{n-*}(\{x\}) \xrightarrow{\;\simeq\;} H_*(X,X\backslash\{x\}) \;\;(x\in X)$$

to the global Poincaré duality isomorphisms

$$[X]\cap - \;:\; H^{n-*}(X) \xrightarrow{\;\simeq\;} H_*(X)\ .$$

The topologically invariant property of the local homology at each point being that of $\mathbb{R}^n$ is assembled to the homotopy invariant property of n-dimensional Poincaré duality.

The *quadratic L-groups* $L_n(R)$ $(n\geq 0)$ of Wall [176] were expressed in Ranicki [141] as the cobordism groups of quadratic Poincaré complexes (C,ψ) over a ring with involution R, with C a f.g. free R-module chain complex and ψ a quadratic structure inducing Poincaré duality isomorphisms $(1+T)\psi_0\colon H^{n-*}(C)\cong H_*(C)$.

The algebraic L-theory assembly map

$$A\;:\;H_*(X;\mathbb{L}_\bullet)\longrightarrow L_*(\mathbb{Z}[\pi_1(X)])$$

is a central feature of the combinatorial theory of surgery, with $H_*(X;\mathbb{L}_\bullet)$ the generalized homology groups of X with coefficients in the 1-connective quadratic L-theory spectrum $\mathbb{L}_\bullet$ of $\mathbb{Z}$. By construction, the structure groups $\mathbb{S}_*(X)$ of a space X are the relative homotopy groups of A, designed to fit into the *algebraic surgery exact sequence*

$$\dots\longrightarrow H_n(X;\mathbb{L}_\bullet)\xrightarrow{\;A\;}L_n(\mathbb{Z}[\pi_1(X)])\xrightarrow{\;\partial\;}\mathbb{S}_n(X)$$
$$\longrightarrow H_{n-1}(X;\mathbb{L}_\bullet)\longrightarrow\dots\ .$$

The structure groups $\mathbb{S}_*(X)$ measure the extent to which the surgery obstruction groups $L_*(\mathbb{Z}[\pi_1(X)])$ fail to be a generalized homology theory, or equivalently the extent to which the algebraic L-theory assembly maps A fail to be isomorphisms. The algebraic surgery exact sequence for a compact manifold M is identified in §18 with the Sullivan–Wall surgery exact sequence for the manifold structure set of M.

The total surgery obstruction $s(X) \in \mathbb{S}_n(X)$ of an n-dimensional Poincaré space X is expressed in §17 in terms of a combinatorial formula measuring the failure on the chain level of the local homology groups $H_*(X, X\backslash\{x\})$ $(x \in X)$ to be isomorphic to $H^{n-*}(\{x\}) = H_*(\mathbb{R}^n, \mathbb{R}^n\backslash\{0\})$. The condition $s(X) = 0$ is equivalent to the cellular $\mathbb{Z}[\pi_1(X)]$-module chain complex $C(\widetilde{X})$ of the universal cover $\widetilde{X}$ being algebraic Poincaré cobordant to the assembly of a local system over X of $\mathbb{Z}$-module chain complexes with Poincaré duality. The structure invariant $s(f) \in \mathbb{S}_{n+1}(M)$ of a homotopy equivalence $f: N \longrightarrow M$ of compact n-dimensional manifolds is expressed in §18 in terms of a combinatorial formula measuring the failure on the chain level of the local homology groups $H_*(f^{-1}(x))$ $(x \in M)$ to be isomorphic to $H_*(\{x\})$. The condition $s(f) = 0$ is equivalent to the algebraic mapping cone $\mathcal{C}(\widetilde{f}: C(\widetilde{N}) \longrightarrow C(\widetilde{M}))_{*+1}$ being algebraic Poincaré cobordant to the assembly of a local system over M of contractible $\mathbb{Z}$-module chain complexes.

The algebraic L-theory assembly map is constructed in §9 as a forgetful map between two algebraic Poincaré bordism theories, in which the underlying chain complexes are the same, but which differ in the duality conditions required. There is a strong 'local' condition and a weak 'global' condition, corresponding to the difference between a manifold and a Poincaré space, and between a homeomorphism and a homotopy equivalence. The assembly of a local algebraic Poincaré complex is a global algebraic Poincaré complex, by analogy with the passage from integral to rational quadratic forms in algebra, and from manifolds to Poincaré spaces in topology. The algebraic L-theory assembly maps have the advantage over the analogous topological assembly maps in that their fibres can be expressed in terms of local algebraic Poincaré complexes such that the underlying chain complexes are globally contractible.

The generalized homology groups of a simplicial complex K with L-theory coefficients are identified in §13 with the cobordism groups of local algebraic Poincaré complexes, where local means that there is a simply connected Poincaré duality condition at each simplex in K. The cobordism groups of global algebraic Poincaré complexes are the surgery obstruction groups or some symmetric analogues, where global means that there is a single non-simply connected Poincaré duality condition over the universal cover

$\widetilde{K}$. Surgery theory identifies the fibre of the assembly map from compact manifolds to finite Poincaré spaces in dimensions ≥ 5 with the fibre of the algebraic L-theory assembly map. Picture this identification as a fibre square

$$\begin{array}{ccc} \{\text{topological manifolds}\} & \longrightarrow & \{\text{local algebraic Poincaré complexes}\} \\ \Big\downarrow \text{assembly} & & \Big\downarrow \text{assembly} \\ \{\text{Poincaré spaces}\} & \longrightarrow & \{\text{global algebraic Poincaré complexes}\} \end{array}$$

allowing the homotopy types of compact manifolds to be created out of the homotopy types of finite Poincaré spaces and some extra chain level Poincaré duality. The assembly maps forget the local structure, and the fibres of the assembly maps measure the difference between the local and global structures. The fibre square substantiates the suggestion of Siebenmann [157, §14] that 'topological manifolds bear the simplest possible relation to their underlying homotopy types'.

The *surgery obstruction* of a normal map $(f,b)\colon M \longrightarrow X$ from a compact n-dimensional manifold M to a finite n-dimensional Poincaré space X

$$\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)])$$

is such that $\sigma_*(f,b) = 0$ if (and for $n \geq 5$ only if) (f,b) is normal bordant to a homotopy equivalence. In the original construction of Wall [176] $\sigma_*(f,b)$ was defined after preliminary geometric surgeries to make (f,b) $[n/2]$-connected. In Ranicki [142] the surgery obstruction was intrepreted as the cobordism class of an n-dimensional quadratic Poincaré complex $(\mathcal{C}(f^!),\psi)$ over $\mathbb{Z}[\pi_1(X)]$ associated directly to (f,b), with

$$f^! : C(\widetilde{X}) \simeq C(\widetilde{X})^{n-*} \xrightarrow{\tilde{f}^*} C(\widetilde{M})^{n-*} \simeq C(\widetilde{M})$$

the Umkehr chain map.

The algebraic Poincaré cobordism approach to the quadratic L-groups $L_*(R)$ extends to n-ads, and hence to the definition of a quadratic $\mathbb{L}$-spectrum $\mathbb{L}.(R)$ with homotopy groups

$$\pi_*(\mathbb{L}.(R)) = L_*(R) .$$

In Ranicki [145] the quadratic L-groups $L_n(\mathbb{A})$ $(n \geq 0)$ of n-dimensional quadratic Poincaré complexes were defined for any additive category with involution $\mathbb{A}$, with

$$L_*(R) = L_*(\mathbb{A}(R)) \quad , \quad \mathbb{A}(R) = \{\text{f.g. free } R\text{-modules}\} .$$

In §1 the quadratic L-groups $L_*(\mathbb{A})$ are defined still more generally, for any additive category $\mathbb{A}$ with a *chain duality*, that is a duality involution on the chain homotopy category.

The chain complex assembly of Ranicki and Weiss [147] provides a convenient framework for dealing with the algebraic L-theory assembly over a simplicial complex K. The method can be extended to arbitrary topological spaces using nerves of open covers.

An (R,K)-*module* M is a f.g. free R-module with a direct sum decomposition

$$M = \sum_{\sigma\in K} M(\sigma)$$

with R a commutative ring. An (R,K)-module morphism $f: M \longrightarrow N$ is an R-module morphism such that

$$f(M(\sigma)) \subseteq \sum_{\tau\geq\sigma} N(\tau) \quad (\sigma\in K) .$$

An (R,K)-module chain complex C is *locally contractible* if it is contractible in the (R,K)-module category, or equivalently if each $C(\sigma)$ $(\sigma\in K)$ is a contractible f.g. R-module chain complex. The *assembly* of an (R,K)-module M is the f.g. free $R[\pi_1(K)]$-module

$$M(\widetilde{K}) = \sum_{\tilde{\sigma}\in\widetilde{K}} M(p(\tilde{\sigma})) ,$$

with $p:\widetilde{K}\longrightarrow K$ the unversal covering projection. An (R,K)-module chain complex C is *globally contractible* if the assembly $C(\widetilde{K})$ is a contractible $R[\pi_1(K)]$-module chain complex. A locally contractible complex is globally contractible, but a globally contractible complex need not be locally contractible.

An n-dimensional quadratic complex (C,ψ) in $\mathbb{A}(R,K)$ is *locally Poincaré* if the algebraic mapping cone of the (R,K)-module chain map $(1+T)\psi_0$: $C^{n-*}\longrightarrow C$ is locally contractible, with each

$$(1+T)\psi_0(\sigma) : C(\sigma)^{n-|\sigma|-*} \longrightarrow C(\sigma)/\partial C(\sigma) \quad (\sigma\in K)$$

an R-module chain equivalence. (See §5 for the construction of the chain duality on $\mathbb{A}(R,K)$.) An n-dimensional quadratic complex (C,ψ) in $\mathbb{A}(R,K)$ is *globally Poincaré* if the algebraic mapping cone of $(1+T)\psi_0: C^{n-*}\longrightarrow C$ is globally contractible, with

$$(1+T)\psi_0 : C^{n-*}(\widetilde{K}) \simeq C(\widetilde{K})^{n-*} \longrightarrow C(\widetilde{K})$$

an $R[\pi_1(K)]$-module chain equivalence. Chain complexes with local (resp. global) Poincaré duality correspond to manifolds (resp. Poincaré spaces).

The generalized homology groups $H_*(K;\mathbb{L}.(R))$ are the cobordism groups of quadratic locally Poincaré complexes in $\mathbb{A}(R,K)$. The algebraic L-theory assembly map

$$A : H_n(K;\mathbb{L}.(R)) \longrightarrow L_n(R[\pi_1(K)]) \; ; \; (C,\psi) \longrightarrow (C(\widetilde{K}),\psi(\widetilde{K}))$$

is defined by forgetting the locally Poincaré structure. The geometric assembly map of Quinn [127], [128], [134] pieces together the non-simply connected surgery obstruction of a normal map of closed manifolds from the simply connected pieces. Similarly, the algebraic L-theory assembly map A pieces together a globally Poincaré complex over $R[\pi_1(K)]$ from a locally Poincaré complex in $\mathbb{A}(R,K)$.

The main algebraic construction of the text is the algebraic surgery exact sequence of §14

$$\dots \longrightarrow H_n(K;\mathbb{L}.(R)) \xrightarrow{A} L_n(R[\pi_1(K)]) \xrightarrow{\partial} \mathbb{S}_n(R,K) \longrightarrow H_{n-1}(K;\mathbb{L}.(R)) \longrightarrow \dots .$$

The *quadratic structure groups* $\mathbb{S}_*(R,K)$ are the cobordism groups of quadratic complexes in $\mathbb{A}(R,K)$ which are locally Poincaré and globally contractible.

The algebraic surgery exact sequence is a generalization of the quadratic L-theory localization exact sequence of Ranicki [143, §3]

$$\dots \longrightarrow L_n(R) \longrightarrow L_n(S^{-1}R) \longrightarrow L_n(R,S) \longrightarrow L_{n-1}(R) \longrightarrow \dots ,$$

for the localization $R \longrightarrow S^{-1}R$ of a ring with involution R inverting a multiplicative subset $S \subset R$ of central non-zero divisors invariant under the involution. The relative L-groups $L_*(R,S)$ are the cobordism groups of quadratic Poincaré complexes (C,ψ) over R such that C is an R-module chain complex with localization $S^{-1}C = S^{-1}R \otimes_R C$ a contractible $S^{-1}R$-module chain complex. In the classic case

$$R = \mathbb{Z} \ , \ S = \mathbb{Z}\backslash\{0\} \ , \ S^{-1}R = \mathbb{Q}$$

the relative L-groups $L_{2i}(R,S)$ are the Witt groups of $\mathbb{Q}/\mathbb{Z}$-valued $(-)^i$-quadratic forms on finite abelian groups, and $L_{2i+1}(R,S) = 0$.

The quadratic structure groups $\mathbb{S}_*(K)$ are defined in §15 as the 1-connective versions of $\mathbb{S}_*(\mathbb{Z},K)$, to fit into the algebraic surgery exact sequence

$$\dots \longrightarrow H_n(K;\mathbb{L}.) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(K)]) \xrightarrow{\partial} \mathbb{S}_n(K) \longrightarrow H_{n-1}(K;\mathbb{L}.) \longrightarrow \dots$$

with $\mathbb{L}.$ the 1-connective cover of $\mathbb{L}.(\mathbb{Z})$. The 0th space $\mathbb{L}_0$ of $\mathbb{L}.$ is homotopy equivalent to the homotopy fibre G/TOP of the forgetful map $BTOP \longrightarrow BG$ from the classifying space for stable topological bundles to the classifying space for stable spherical fibrations. The homotopy groups of $\mathbb{L}.$ are the simply connected surgery obstruction groups

$$\pi_n(\mathbb{L}.) = \pi_n(G/TOP) = L_n(\mathbb{Z}) = \begin{cases} \mathbb{Z} \\ 0 \\ \mathbb{Z}_2 \\ 0 \end{cases} \text{if } n \equiv \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \pmod 4 .$$

The *dual cells* of a simplicial complex K are the subcomplexes of the barycentric subdivision K' defined by

$$D(\sigma, K) \;=\; \{\,\widehat{\sigma}_0\widehat{\sigma}_1\ldots\widehat{\sigma}_r \in K' \,|\, \sigma \leq \sigma_0 < \sigma_1 < \ldots < \sigma_r\,\}\ ,$$

with boundary

$$\partial D(\sigma, K) \;=\; \bigcup_{\tau > \sigma} D(\tau, K)\ .$$

Transversality is functorial in the PL category: Cohen [36] proved that for a simplicial map $f: M \longrightarrow K'$ from a compact n-dimensional PL manifold M the inverse images of the dual cells

$$(M(\sigma), \partial M(\sigma)) \;=\; f^{-1}(D(\sigma, K), \partial D(\sigma, K)) \quad (\sigma \in K)$$

are $(n - |\sigma|)$-dimensional PL manifolds with boundary. An abstract version of this transversality is used in §12 to express the groups $h_*(K)$ for any generalized homology theory as the cobordism groups of 'h-cycles in K'', which are compatible assignations at each simplex $\sigma \in K$ of a piece of the coefficient group $h_*(\{\text{pt.}\})$. This is the combinatorial analogue of the result that every generalized homology theory is the cobordism of compact manifolds with singularities of a prescribed type (Sullivan [166], Buoncristiano, Rourke and Sanderson [21]).

A *finite n-dimensional geometric Poincaré complex* X is a finite simplicial complex such that the polyhedron is an n-dimensional Poincaré space. The total surgery obstruction of X is defined in §17 to be the cobordism class

$$s(X) \;=\; (\Gamma, \psi) \in \mathbb{S}_n(X)$$

of an $(n-1)$-dimensional quadratic locally Poincaré globally contractible complex (Γ, ψ) in $\mathbb{A}(\mathbb{Z}, X)$ with

$$\begin{aligned} &H_*(\Gamma(\sigma)) \\ &\quad= H_{*+1}(\phi(\sigma) : C(D(\sigma, X))^{n-|\sigma|-*} \longrightarrow C(D(\sigma, X), \partial D(\sigma, X))) \\ &\quad= H_{*+|\sigma|+1}([X]_x \cap - : C(\{x\})^{n-*} \longrightarrow C(X, X\backslash\{x\})) \end{aligned}$$

measuring the failure of local Poincaré duality at the barycentre $x = \hat{\sigma} \in X$ of each simplex $\sigma \in X$. The assembly $(n-1)$-dimensional quadratic Poincaré complex $(\Gamma(\widetilde{X}), \psi(\widetilde{X}))$ over $\mathbb{Z}[\pi_1(X)]$ is contractible, with

$$\Gamma(\widetilde{X}) \;=\; \mathcal{C}([X] \cap - : C(\widetilde{X})^{n-*} \longrightarrow C(\widetilde{X}))_{*+1} \;\simeq\; 0\ .$$

The structure invariant $s(f) \in \mathbb{S}_{n+1}(M)$ of a homotopy equivalence $f: N \longrightarrow M$ of closed n-dimensional manifolds is defined in §18, measuring the extent up to algebraic Poincaré cobordism to which the point inverses $f^{-1}(x)$ are contractible. The invariant is such that $s(f) = 0$ if (and for $n \geq 5$ only if) f is h-cobordant to a homeomorphism. The total surgery obstruction has the following interpretation: for $n \geq 5$ a finite n-dimensional Poincaré space X is homotopy equivalent to a compact topological manifold if and only if the Poincaré duality chain equivalence has 'contractible point-inverses' up to an appropriate cobordism relation.

The *structure set* $\mathbb{S}^{TOP}(X)$ of an n-dimensional Poincaré space X is the set (possibly empty) of h-cobordism classes of pairs

$$(\text{compact } n\text{-dimensional topological manifold } M\,, \quad \text{homotopy equivalence } f: M \longrightarrow X)\ .$$

The structure set of a compact manifold M is non-empty, with base point $(M,1) \in \mathbb{S}^{TOP}(M)$.

The structure invariant $s(f) \in \mathbb{S}_{n+1}(M)$ of a homotopy equivalence of compact n-dimensional manifolds $f: N \longrightarrow M$ is defined in §18 to be the cobordism class

$$s(f) \;=\; (\Gamma, \psi) \in \mathbb{S}_{n+1}(M)$$

of an n-dimensional quadratic locally Poincaré complex (Γ, ψ) in $\mathbb{A}(\mathbb{Z}, M)$ with contractible assembly

$$\Gamma(\widetilde{M}) \;=\; \mathcal{C}(\tilde{f}: C(\widetilde{N}) \longrightarrow C(\widetilde{M})) \;\simeq\; 0\ .$$

The $\mathbb{Z}$-module chain complexes $\Gamma(\sigma)$ $(\sigma \in M)$ are the quadratic Poincaré kernels of the normal maps of $(n-|\sigma|)$-dimensional manifolds with boundary

$$f| \;:\; (gf)^{-1}D(\sigma, M) \longrightarrow g^{-1}D(\sigma, M) \quad (\sigma \in M)\ .$$

(For the sake of convenience it is assumed here that M is the polyhedron of a finite simplicial complex, but this assumption is avoided in §18). The structure invariant can also be viewed as the rel ∂ total surgery obstruction

$$s(f) \;=\; s_\partial(W, N \sqcup -M) \in \mathbb{S}_{n+1}(W) \;=\; \mathbb{S}_{n+1}(M)$$

with $(W, N \sqcup -M)$ the finite $(n+1)$-dimensional Poincaré pair with manifold boundary defined by the mapping cylinder $W = N \times I \cup_f M$.

The Sullivan–Wall geometric surgery exact sequence of pointed sets for a compact n-dimensional manifold M with $n \geq 5$

$$\dots \longrightarrow L_{n+1}(\mathbb{Z}[\pi_1(M)]) \longrightarrow \mathbb{S}^{TOP}(M) \longrightarrow [M, G/TOP] \longrightarrow L_n(\mathbb{Z}[\pi_1(M)])$$

is shown in §18 to be isomorphic to the 1-connective algebraic surgery exact sequence of abelian groups

$$\dots \longrightarrow L_{n+1}(\mathbb{Z}[\pi_1(M)]) \xrightarrow{\partial} \mathbb{S}_{n+1}(M) \longrightarrow H_n(M; \mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(M)])\ .$$

The function sending a homotopy equivalence of manifolds to its quadratic structure invariant defines a bijection

$$s \;:\; \mathbb{S}^{TOP}(M) \longrightarrow \mathbb{S}_{n+1}(M)\ ;\ f \longrightarrow s(f)$$

between the manifold structure set and the quadratic structure group.