Introduction

This book is based on a series of lectures (the so called 'Lezioni Lincee') delivered by the author in the Accademia dei Lincei, Rome University, the Scuola Normale Superiore (Pisa) and Pavia University.

The main attention is given to the global asymptotics of solutions $u(t) = u(t,x)$ of evolutionary equations

$$\partial_t u = A(u), \quad u\,|_{t=0} = u_0. \tag{1}$$

The main results are illustrated by examples of the Navier-Stokes equations, the system of reaction-diffusion equations, the semilinear wave equation with dissipation, and other systems of parabolic and hyperbolic equations.

In the Appendix non-autonomous dynamical systems are studied. We assume that the dependence on time is quasi-periodic or almost periodic. We investigate problems related to the existence and dimension of attractors.

Let problem (1) in the autonomous case have a unique solution $u(t)$, $t \geq 0$, and let $u(t)$ belong to the same Banach space E for each t. In this case to system (1) there corresponds a semigroup of operators $\{S_t \mid t \geq 0\}$, $S_t : E \to E$, where $S_t u_0 = u(t)$ and $u(t)$ is the solution of (1). So we shall study the global behaviour of solutions mainly in terms of this semigroup.

The book includes the results obtained by the author in collaboration with A. V. Babin, M. Y. Skvortsov, V. Y. Skvortsov and V. V. Chepyzhov.

In the first chapter we remind the reader of the definitions relating to the maximal (global) attactor of a semigroup $\{S_t\}$ and formulate the existence theorem. We also give the definition of an invariant manifold containing an equilibrium point z of a semigroup $\{S_t\}$ and state its main properties.

In the second chapter the spectral asymptotics are described in the simplest case when the solution $u(t)$ of (1) tends to an equilibrium point z (not necessarily stable) as $t \to +\infty$.

We note that when equation (1) is linear and the solution can be found as a Fourier series, then the spectral asymptotics of $u(t)$ coincides with the partial sum $u_N(t)$ of its Fourier series. Let λ_k be the k-th eigenvalue of the operator A. Then the inequality

$$||u(t) - u_N(t)|| \leq Ce^{-\lambda_{N+1}t} \tag{2}$$

holds. For simplicity we assume that A is a self-adjoint operator semi-bounded from above and that it has a discrete spectrum $\{-\lambda_n\}$, where $\lambda_n \to +\infty$ for $n \to \infty$.

Asymptotics similar to (2) also exist when the operator $A(u)$ is non-linear and $u(t) \to z$, $t \to +\infty$, where z is an equilibrium point of the semigroup $\{S_t\}$.

We assume that the operators S_t have the Fréchet derivative $S_t'(z)$ at the point z. Obviously, the set of the operators $S_t'(z)$ for $t \geq 0$ form a linear semigroup. If the circle $|\zeta| = \rho$ does not intersect the spectrum of the operator $S_1'(z)$, we call ρ a regular spectral radius. To the part of spectrum lying outside this circle there corresponds an invariant subspace of $S_1'(z)$ (it is also invariant with respect to $S_t'(z)$ for $t \geq 0$). We denote this space by $E_+(z,\rho)$ and assume that $E_+(z,\rho)$ is finite-dimensional. If some natural conditions are satisfied, then there exists a finite-dimensional $M_+(z,\rho)$, which is locally invariant with respect to $\{S_t\}$ and is tangent to $E_+(z,\rho)$ at the point z.

In §3 it is proved that if $u(t) \to z$ when $t \to +\infty$ and ρ is a regular spectral radius, then on the manifold $M_+(z,\rho)$ there exists a trajectory $\tilde{u}(t)$ (spectral asymptotic for $u(t)$) such that

$$||u(t) - \tilde{u}(t)|| \leq C\rho^t \quad \forall t \geq T. \tag{3}$$

In the case of nonlinear parabolic equations or the Navier-Stokes system we can take a sequence of regular radii $\rho_j \to 0$, and then the corresponding sequence $\tilde{u}_j \in M_+(z,\rho_j)$ will approximate $u(t)$ with increasing accuracy (Babin & Vishik [1], [7]).

We note that the rate of approach of $u(t)$ to the equilibrium point z was studied by Foias & Saut [1], Foias & Guillopé [1], Haraux [1], [2], and other authors. The relationship between the trajectories belonging to $M_+(z,\rho)$ $(\rho = 1 - \varepsilon)$ and arbitrary trajectories belonging to a neighbourhood of the point z was studied by Pliss [1]. These results were applied to the problem of stability or instability of the equilibrium point z of ordinary differential equations system.

In Chapter III uniform asymptotics (with respect to the initial data u_0, when u_0 belongs to a bounded set $B \subseteq E$) is constructed. It is assumed that the semigroup $\{S_t\}$ has a global Lyapunov function P, which decreases along all trajectories $S_t u_0$ except for the equilibrium points z. We also assume that the set $\mathcal{M}$ of equilibrium points z is finite, $\mathcal{M} = (z_1, ..., z_N)$. In this case uniform asymptotics of the trajectories $u(t)$ are described in terms of piecewise continuous trajectories lying on $\cup_j M^+(z_j, \rho_j)$, where $M^+(z_j, \rho_j) = \cup_{t\geq 0} S_t M_+(z_j, \rho_j)$. Each continuous part of the asymptotic $\tilde{u}(t)$ lies on one of the manifolds $M_+(z_j, \rho_j)$, and the values of $\tilde{u}(t)$ at its discontinunity points, lie in a small neighbourhood of an unstable equilibrium point $z_j \in \mathcal{M}$. The number of such points does not exceed N. Obviously, the trajectories $\tilde{u}(t)$ form a finitely-parametrized family of spectral asymptotics. In Chapter V for arbitrary trajectory $u(t) = S_t u_0$, $u_0 \in B$, a spectral asymptotic $\tilde{u}(t)$ is constructed such that

$$||u(t) - \tilde{u}(t)|| \leq Ce^{-\eta t} \quad \forall t \geq 0, \tag{4}$$

where the constant C depends only on B and $\eta > 0$ depends on spectral radii ρ_j $(j = 1, ..., N)$. If all $\rho_j \to 0$, then $\eta \to +\infty$ (Babin & Vishik [1], [4], [7]).

The relationship between the above constructions and the ideas of Fourier, Lyapunov and Poincaré is quite clear.

We also note that although the dynamic system corresponding to the semigroup $\{S_t\}$ is considered in the infinite-dimensional space E, its asymptotics are described by means of a finite-dimensional dynamical system generated by the semigroup $\{S_t\}$ on the finite-dimensional manifolds $M^+(z_j, \rho_j)$.

In §§4 and 5 we give examples of equations and systems having local and uniform spectral asymptotics. Among these examples are various equations from mathematical physics such as the two- and three-dimensional Navier-Stokes systems, the reaction-diffusion system, quasilinear parabolic equations, and a hyperbolic equation with dissipation.

In §6 of Chapter IV we study perturbation theory for equations with a parameter,

$$\partial_t u = A(u, \lambda), \quad u_{t=0} = u_0 \quad (|\lambda| \leq \lambda_0). \tag{5}$$

Naturally, the corresponding semigroup also depends on λ, i.e. $S_t = S_t(\lambda)$.

A number of results on the behaviour of the solutions $u(t, \lambda)$ for large t were given by Hale [2], Mora & Sola-Morales [1] and other authors.

We state the following results.

Let $u = u(t, \lambda)$ be a solution to problem (5) and let $v(t) = S_t(0)u_0$ be the zeroth term of the λ-asymptotics, i.e., a solution of (5) for $\lambda = 0$. Then, under some natural conditions on $A(u, \lambda)$, the following estimate holds:

$$||u(t,\lambda) - v(t)|| \leq C|\lambda|e^{\alpha t}, \quad \alpha > 0. \tag{6}$$

It is clear that with increasing t the estimate (6) worsens. Let us suppose that the operator $S_t(0)$ satisfies the conditions for the existence of uniform spectral asymptotics in the case when all $\rho_j = 1$ and the following inequality holds:

$$||S_t(\lambda)u_0 - S_t(0)v_0|| \leq Ce^{\alpha t}(|\lambda|^\gamma + ||u_0 - v_0||), \quad \gamma > 0. \tag{7}$$

In this case the above function $v(t) = S_t(0)u_0$ can be stabilised for $t \to +\infty$. More precisely, for each $u(t) = S_t(\lambda)u_0$, where $u_0 \in B$ (B is a bounded set in E), there exists a trajectory $\tilde{u}(t)$ such that

$$\tilde{u}(t) = v(t) = S_t(0)u_0$$

for $t \in [0, T(u)]$ and

$$\tilde{u}(t) \in \bigcup_j M^+(z_j, 1)$$

for $t > T$. As above, for $t > T$ the approximation $\tilde{u}(t)$ is piecewise continuous with respect to t, and $\tilde{u}(t)$ belongs to a finitely-parametrized family of trajectories $\{S_t(0)\}$. The continuous parts of $\tilde{u}(t)$ lie on the manifolds $M^+(z_j, 1)$. It will be shown (in §6) that $\tilde{u}(t)$ can be chosen so that the estimate

$$\sup_{t \geq 0} ||u(t,\lambda) - \tilde{u}(t)|| \leq C|\lambda|^q$$

holds uniformly with respect to u_0 when u_0 belongs to a bounded set B. Here $C = C(B)$ and $q > 0$ depends on the spectral characteristic of operators $S_1'(z_j)$ (Babin & Vishik [1], [4], [7]).

The stabilised asymptotics are illustrated in §7 by examples of the reaction-diffusion system, hyperbolic systems with dissipation, and parabolic systems of equations with a small parameter in spatial derivatives of the highest order.

In the second, third and fourth chapters we give a summary of some results obtained by Babin & Vishik [1], [4], [6], [7].

In Chapter V we describe stabilized asymptotics for the reaction-diffusion system with a small parameter in the time derivative:

$$\left.\begin{aligned} \varepsilon\partial_t u_1 &= \Delta u_1 - f_1(u_1,u_2) - g_1(x), \quad x \in \Omega \subseteq \mathbf{R}^n, \\ \partial_t u_2 &= \Delta u_2 - f_2(u_1,u_2) - g_2(x), \\ u_1 |_{\partial\Omega} = 0, \quad & u_1 |_{\partial\Omega} = 0 \text{ or } \frac{\partial u_1}{\partial\nu} |_{\partial\Omega} = 0, \quad \frac{\partial u_2}{\partial\nu} |_{\partial\Omega} = 0, \end{aligned}\right\} \tag{8}$$

and initial conditions

$$u_1 |_{t=0} = u_1^0, \quad u_2 |_{t=0} = u_2^0.$$

We note that when $\varepsilon = 0$, the first equation of this system becomes stationary. The semigroup $\{S_t(\varepsilon)\}$ that corresponds to system (8) acts in the phase space $E = H_1(\Omega) \times H_1(\Omega)$, and for $\varepsilon = 0$ it turns into the semigroup $\{S_t(0)\}$ which acts in another phase space $E_1 = H_1(\Omega)$. This action ($\varepsilon = 0$) can be determined, if we find $u_1 = u_1(u_2)$ from the first (stationary) equation (with $\varepsilon = 0$) in (8), substitute it into the second equation, and then find $S_t u_2(0) = u_2(t)$, where $u_2(t)$ is the solution of the second equation with the initial data $u_2(0)$.

When $f_1(u_1,u_2)$ and $f_2(u_1,u_2)$ satisfy some conditions and the initial functions are bounded, we can construct stabilised asymptotics

$$(v_1(t), v_2(t))$$

such that

$$\left.\begin{aligned} \sup_{t\geq 0} \|u_2(t,\varepsilon) - v_2(t)\| \leq C\varepsilon^q, \\ \sup_{t\geq \tau} \|u_1(t,\varepsilon) - v_1(t)\| \leq C\varepsilon^q. \end{aligned}\right\} q > 0, \tau > 0. \tag{9}$$

Here the functions $v_1(t)$, $v_2(t)$ are piecewise continuous, each continuous part of $v_1(t)$, $v_2(t)$ is a solution of (8) for $\varepsilon = 0$, and each continuous part of $v_2(t)$ (except the first part) lies on a finite-dimensional unstable manifold of the semigroup $\{S_t(0)\}$ (Vishik & V. Skvortsov [1], [2]).

Problems with a small parameter in spatial derivatives of the highest order are studied in Chapters VI and VII.

A characteristic feature of these problems is the presence of a boundary layer term in the asymptotics. For simplicity, we consider a parabolic equation

$$\frac{\partial u}{\partial t} = -\varepsilon^2 \Delta^2 u + \Delta u - f(u) - g(x), \tag{10}$$

$$u |_{\partial\Omega} = 0, \quad \frac{\partial u}{\partial\nu} |_{\partial\Omega} = 0, \quad x \in \Omega \subseteq \mathbf{R}^n. \tag{11}$$

For $\varepsilon = 0$ the equation (10) is a parabolic equation of the second order, and we can make it satisfy only the first boundary condition.

We study an asymptotic expansion of the elements $u(x)$ lying on the attractor $\mathcal{A}^\varepsilon$ for the problem (10), (11). Under some conditions on $f(u)$ the following expansion holds:

$$u(x) = (u_0 + v_0 - \varepsilon c_0) + \varepsilon(u_1 + v_1 - \varepsilon c_1) + r, \tag{12}$$

where $u_0(x)$ and $u_1(x)$ have uniformly bounded norms with respect to $u(x) \in \mathcal{A}^\varepsilon$ in $H_3(\Omega)$. The functions v_0 and v_1 are of the form of a boundary layer, that is in the neighbourhood of $\partial\Omega$ we have $v_i(x) = \varepsilon c_i(x')e^{-\rho/\varepsilon}$, where $x' \in \partial\Omega$ and ρ is the distance along the normal at x' from this point to x and $c_0(x)$ and $c_1(x)$ are smooth functions.

The remainder r admits of the estimate

$$||r||_{H_3(\Omega)} + \varepsilon^{-1}||r||_{H_2(\Omega)} + \varepsilon^{-2}||r||_{H_1(\Omega)} \leq M, \tag{13}$$

where the constant M does not depend on ε and on $u \in \mathcal{A}^\varepsilon$ (M. Skvortsov & Vishik [1]). Let us note that the function $u(x)$ has a bounded norm only in $H_{3/2}(\Omega)$, and the index $3/2$ cannot be increased (see M. Skvortsov [1], [2], [3]). The 'boundary layer' functions v_0 and v_1 have bounded norms in $H_{3/2}(\Omega)$ uniformly with respect to ε for $0 < \varepsilon \leq \varepsilon_0$. Obviously, these functions are quite essential in the expansion (12).

Formula (12) is similar to the asymptotic expansion of the solution of an elliptic equation with small parameter in the highest derivatives, which was studied by Vishik & Lusternik [1], [2].

Denote by $U(R)$ the set of trajectories $\{u(t)\}$ of equation (10) with bounded initial data, $||u(t_0)|| \leq R$, $t_0 \in \mathbf{R}$ fixed. For any element $u(T)$, for fixed $T > t_0$ and $u(t) \in U(R)$, the following asymptotic expansion is established:

$$u(T) = u_0 + v + r. \tag{14}$$

Here the function u_0 is bounded in $H_3(\Omega)$ uniformly with respect to ε and for $u(t) \in U(R)$, v is the first-order boundary layer function (see Vishik & Lusternik [1], [2]), and r is the remainder.

From the expansion (14) we derive in §16 the asymptotics of the trajectory $u(t)$ on any finite time interval $t \in [t_1, t_1 + T_1]$, $t_1 \in \mathbf{R}$, $T_1 > 0$:

$$u(t) = w(t) + v(t) + r(t) \tag{15}$$

(Vishik & M. Skvortsov [2]). Here $w(t)$ is the solution of the limit (for $\varepsilon = 0$) parabolic equation

$$\partial_t w = \Delta w - f(w) - g, \quad w\,|_{\partial\Omega} = 0 \tag{16}$$

with initial data $w(t_1) = u_0$, where u_0 is defined in (14) for $T = t_1$, $v(t)$ is the boundary layer function of the first order, and $r(t)$ is the remainder.

Let the set $\mathcal{M} = (z_1, ..., z_N)$ of equilibrium points of (16) be finite. Let $\delta > 0$ and let the trajectory $u(t)$ belonging to $U(R)$ for $t \in (t_1, t_2)$ be outside the δ-neighbourhood $\mathcal{O}_\delta(\mathcal{M})$. Then under certain conditions on f and g, (i) the time $T = t_2 - t_1$, for which $u(t)$ is outside $O_\delta(\mathcal{M})$, is bounded uniformly for $\varepsilon \in (0, \varepsilon_0]$ and $u(t) \in U(R)$, (ii) the asymptotic expansion (15) of the trajectory $u(t)$ for $t \in [t_1, t_2]$ is valid and the remainder $r(t)$ satisfies the following inequality:

$$\varepsilon^2 ||r(t)||_2^2 + ||r(t)||_1^2 + \int_{t_1}^{t_2} ||\partial_t r||^2 \, d\tau \leq C\varepsilon^2, \tag{17}$$

where $C = C(\delta, R)$ is independent of ε and of $u(t)$ (M. Vishik & M. Skvortsov [2]).

In §§15 and 17 similar results are established for the set of solutions of equation (10) with boundary conditions

$$u \mid_{\partial\Omega} = 0, \quad \Delta u \mid_{\partial\Omega} = 0. \tag{18}$$

The global asymptotics of the trajectories of problem (10), (18) for any $t \geq t_0$ is given. It is similar to (15) if $u(t) \notin \mathcal{O}_\delta(\mathcal{M})$, and estimate (17) holds. Inside the neighbourhood $\mathcal{O}^0_{\delta_0}(\mathcal{M})$ (in the metric of H) only the principal asymptotic term is constructed (the stabilized asymptotics as in Chapter IV). It consists of parts of trajectories $w(t)$ of the limit equation.

The results obtained in works by M. Skvortsov and M. Vishik are given in the sixth and seventh chapters.

In the Appendix we present the results of works of V. Chepyzhov and M. Vishik. It contains the study of non-autonomous infinite-dimensional dynamical systems generated by evolutionary equations:

$$\left.\begin{aligned} \partial_t u &= A(u,t), \quad t \geq \tau, \\ u \mid_{t=\tau} &= u_\tau, \quad \tau \in \mathbf{R}, \quad u_\tau \in E. \end{aligned}\right\} \tag{19}$$

Here E is a Banach space. We assume that the nonlinear operator $A(u,t)$ depends on time, and that this dependence is quasi-periodic or almost periodic. We consider the following problems of mathematical physics as examples of (19).

1. The two-dimensional Navier-Stokes system:

$$\begin{gathered} \partial_t u = -vLu - B(u,u) + g(x,t), \\ u = (u^1, u^2), \quad u \mid_{\partial\Omega} = 0, \quad x \in \Omega \Subset \mathbf{R}^2, \end{gathered} \tag{20}$$

where

$$L = \Pi\Delta, \quad B(u,u) = \Pi \sum_{i=1}^{2} (u^i \partial_{x_i} u^i)$$

where Π is the projection to the subspace of vector functions with a free divergence.

2. The reaction-diffusion system:

$$\partial_t u = a\Delta u - f(u,x,t) + g(x,t), \tag{21}$$
$$u\,|_{\partial\Omega} = 0 \text{ or } \frac{\partial u}{\partial v}\,|_{\partial\Omega} = 0,$$

where $u = (u^1, ..., u^n)$, $f = (f^1, ..., f^n)$, $g = (g^1, ..., g^n)$, $x \in \Omega \Subset \mathbf{R}^n$, and the matrix $a + a^* > 0$ and the function f satisfies some natural conditions.

3. The hyperbolic equation with dissipation:

$$\partial_t u^2 + \gamma\partial_t u = \Delta u - f(u,x,t) + g(x,t), \tag{22}$$
$$u\,|_{\partial\Omega} = 0, \quad \gamma > 0,$$

which occurs in quantum field theory.

We state some new results concerning the existence of attractors for the above problems and estimate the dimension of these attractors.

If problem (19) is well posed and $u(t) \in E$, $\forall t \geq \tau$, then the solution $u(t)$ can be represented in the form $u(t) = U(t,\tau)u_\tau$, where $U(t,\tau) : E \to E$, $t \geq \tau$, $\tau \in \mathbf{R}$. The family of maps $\{U(t,\tau) \mid E \to E,\ t \geq \tau,\ \tau \in \mathbf{R}\}$ is called the process on E generated by problem (19). By analogy with the autonomous case, we introduce the concepts of absorbing and attracting set for the process $\{U(t,\tau)\}$. The closed set $\mathcal{A}_1 \subseteq E$ is called the attractor of the process $\{U(t,\tau)\}$ if it is attracting ($\text{dist}_E(U(t,\tau)B, \mathcal{A}_1) \to 0$, $t - \tau \to +\infty$, where B is an arbitrary bounded set in E) and satisfies the minimal property. The minimal property is the generalization of the invariance property in the definition of an attractor of a semigroup. (See Sell [1], [2], Dafermos [2], [3], Haraux [3], where the main notions were introduced and some important results concerning attractors of processes were established).

In the Appendix we consider a family of processes $\{U_G(t,\tau)\}$ depending on some functional parameter G called the time symbol of a process. This parameter G is determined by the right-hand side g and, in examples (21), (22), by the nonlinear function f. We prove some general theorems on the existence of a uniform attractor of a family of processes with almost periodic time symbol. We describe a simple method for constructing such attractors. Let, for example, the operator $A(u,t)$ be quasi-periodic with respect to t, i.e.

$$A(u,t) = A_1(u, \omega_1, ..., \omega_k) = A_1(x, \omega)$$

where A_1 is 2π-periodic with respect to each ω_i, $\omega = \Lambda t + \omega_0 \in T^k$ ($\omega_i = \lambda_i t + \omega_{i0}$, $i = 1, ..., k$), $\Lambda = (\lambda_1, ..., \lambda_k)$, $\lambda_1, ..., \lambda_k$ are rationally independent real numbers, and T^k is a k-dimensional torus. Then equation (1) (or equations (20), (21), (22)) can be reduced to the autonomous system

$$\left.\begin{array}{l} \partial_t u = A_1(u,\omega), \quad u\mid_{t=0} = u_0, \\ \partial_t \omega = \Lambda, \quad \omega \mid_{t=0} = \omega_0. \end{array}\right\} \tag{23}$$

System (23) generates the semigroup $\{S(t)\}$ acting on $E \times T^k$ according to the formula

$$S(t)(u_0, \omega_0) = (u(t), \Lambda t + \omega) \quad \forall t \geq 0,$$
$$S(t) : E \times T^k \quad \forall t \geq 0.$$

If the semigroup $\{S(t)\}$ has a compact absorbing or attracting set and is continuous, then the attractor $\mathcal{A} \Subset E \times T^k$ of the semigroup $\{S(t)\}$ exists. The projection $\Pi_1 \mathcal{A} = \mathcal{A}_1 \Subset E$, where Π_1 is the projection to the first component u of the pair (u, ω), is the uniform attractor with respect to $\omega_0 \in T^k$ of the process $\{U_{\omega_0}(t,\tau)\}$ generated by system (19) in the quasi-periodic case. It is shown in the Appendix that problems (20), (21), (22) possess uniform attractors $\mathcal{A}_1$ when the functions $g(x,t)$ and $f(u,x,t)$ are quasi-periodic or almost periodic.

Note that if $g(t) = g(x,t)$ is an almost periodic function, then to define the semigroup $\{S(t)\}$ one must use, instead of T^k, the hull $H(g)$ of the almost periodic function g. Problems (21) and (22) use analogous constructions for almost periodic functions $f(u,x,t)$.

Using this technique we obtain upper bounds for the Hausdorff dimension of attractors for problems (20) and (21) in the quasi-periodic case. For instance, we prove the following estimate for the Hausdorff dimension of the attractor $\mathcal{A}_1$ of the Navier-Stokes system (20):

$$\dim_H \mathcal{A}_1 \leq k + Ck^{1/3} + \mathcal{O}(1). \tag{24}$$

Here we assume, for simplicity, that the Reynolds numbers are bounded ($\nu = \mathcal{O}(1), ||g|| \leq C_1$). Thus, if the number k of rational independent frequencies is increasing, then the upper bound for the dimension of the attractor $\mathcal{A}_1$ also increases. Examples show that estimate (24) is exact.

In the last section of the Appendix we give upper bounds for the Hausdorff dimension of sections of attractors $\mathcal{A}_1$ when g and f are fixed and time t is also fixed.

Let $X = X_{g,f} = \{u(t)\}$ be a family of all bounded solutions for $t \in \mathbb{R}$ to the equation (20), (21) or (22) with given external force g and nonlinear function f (in the case of (20) or (21)). The section $X(t_0)$ at time t_0 is the union of all $u(t_0)$ for $u(t) \in X$. It is shown that the Hausdorff dimension of $X(t)$ is finite. For instance, for the Navier-Stokes system we have

$$\dim_H X(t_0) \leq \frac{C}{\nu^2} \quad \forall t_0 \in \mathbb{R}.$$

This estimate agrees with the corresponding result for the attractor dimension in the autonomous case.