Mathematicians have been fascinated by the geometry of curves and surfaces ever since the time of Newton, one of the most powerful tools used being the calculus. Today's singularity theory is a direct and exceptionally versatile descendant of the differential calculus. In this book the authors introduce this new tool and illustrate its power in a wide variety of very concrete geometrical situations. Only some linear algebra and calculus of functions of several variables are assumed as prerequisites.

The second edition has been thoroughly revised and includes a large number of new applications, examples and exercises. A major new feature is the final chapter, which introduces the reader to the topic, which has central importance in the theory, of classification of functions of several variables. A self-contained account is given, more detailed and elementary than those available elsewhere, of the infinitesimal methods used in this important area.

Curves and singularities

'Singularity is almost invariably a clue.'

Curves and singularities

A geometrical introduction to singularity theory

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Second edition

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For Linda and Rachel

Contents

Preface to the second edition

> 'Have you the effrontery necessary to put it through?' . . .
> 'We can but try – the motto of the firm.'
> (*The Creeping Man*)

We have made a substantial number of small additions to the text and a small number of substantial ones. The small additions are usually in the form of extra exercises and applications which have occurred to us, or have otherwise come to our attention, since the first edition went to press. For example, in chapter 7 we give a pleasant application of caustics and orthotomics to seismology and a discussion of developable surfaces which generalizes the previous material on the developable of a space curve. Our criterion for the inclusion of these extras has always been that they should further illuminate the theory in a way that is geometrically satisfying.

There are several more substantial additions, some of which are complete expositions of topics, while others are sketches intended to whet the reader's appetite for more advanced treatments. An example of the former is the discussion of the Morse lemma in chapter 4. It seemed to us worth while to include, in the first half of the book, a full treatment of the simplest case of classification of a function of two or more variables, namely the case of nondegenerate or Morse functions. This has a number of striking geometrical applications, and some of these are given as exercises. We also sketch the more powerful Morse lemma with parameters. In chapter 6 we have extended the discussion of cusps to include a sketch of the classification methods used for maps from the line to the plane under changes of coordinates in both line and plane (so-called $\mathscr{A}$-equivalence or right–left equivalence). The most significant modification is the replacement of chapter 11 by a substantial discussion of the classification of functions of many variables (i.e. more than the one variable treated in chapter 3). We hope that this new chapter provides a more accessible entry to this important area than was previously available.

This new material also introduces the last basic technique in singularity theory, which is usually known as John Mather's Yoga of Vector Fields. This should help to bridge the gap between our text and more advanced books and research papers.

A Russian translation of the first edition appeared in 1988 (Mir publishers, Moscow), with an introduction by V. I. Arnold in which he gave some historical background to the subject (material which, as he said, was lacking in our book), together with a brief explanation of how the material of the book fits into the more general context of Lagrangean and Legendrian manifolds and maps (a theory with which the Russian school in Moscow is closely associated). We have attempted to remedy the lack of historical background by a brief historical note placed at the end of this edition. But we are conscious of the perils of such an exercise, and we have only mentioned the most obvious antecedents of the material we present in the book.

We are grateful to several people for their help and encouragement; not least Professor Arnold for arranging that our book be translated into Russian and for writing a characteristically acerbic introduction. We also thank David Tranah of Cambridge University Press for encouraging us to write a second edition, and Tim Wilkinson for redrawing a number of the figures. The first author is grateful to Five Colleges, Inc., of Amherst, Massachusetts, and particularly to Professor Donal O'Shea of Mount Holyoke College, for support and hospitality during the period of preparing this edition.

J. W. Bruce

November 1991 P. J. Giblin

Preface to the first edition

> 'I don't think you need alarm yourself,' said I. 'I have usually found that there is method in his madness.' 'Some folk might say there was madness in his method,' muttered the Inspector.
> (*The Memoirs of Sherlock Holmes*)

The object of this book is to introduce to a new generation of students an area of mathematics that has received a tremendous impetus during the last twenty years or so from developments in singularity theory.

The differential geometry of curves, families of curves and surfaces in Euclidean space has fascinated mathematicians and users of mathematics since Newton's time. A minor revolution in mathematical thought and technique occurred during the 1960s, largely through the inventive genius of the French mathematician René Thom. His ideas (partly inspired by the earlier researches of H. Whitney) gave birth to what is now called singularity theory, a term which includes catastrophes and bifurcations. Not only has singularity theory made precise sense of what many of the earlier writers on differential geometry were groping to say (as so often happens, their instinct was uncannily good but they lacked the proper formal setting for their ideas); it has also made possible a richness of detail that would have stirred the imagination of any of the great geometers of the past.

Thom applied his ideas to many fields besides geometry, for example in his famous (but very difficult) book (Thom, 1975). Since then he and others have sharpened and modified the ideas so that the applications to science, potential and actual, are now very impressive – also occasionally very controversial. We believe that the best way to introduce singularity theory is to show it in action in a situation where there is no doubt about what is going on and which requires a minimum of specialist knowledge. For that reason we have included only geometrical applications here, apart from a brief look at catastrophe machines in chapter 1, and some geometric optics and differential equations in chapter 7. Further, in order to keep the technicalities under control, we have restricted ourselves to geometrical applications of singularity theory that hinge on the concepts of transversality and of unfolding a smooth function of one variable. Thus we include the geometry of plane and space curves, envelopes of one-parameter families of curves and surfaces, evolutes, duals, contact, apparent contours of surfaces, caustics, symmetry sets, maps from the plane to the plane and the 'generic geometry' of plane and space curves – properties which hold for 'almost all' curves. (Even if these topics do not mean much to you, we hope that a casual glance through the diagrams will convince you that they are potentially interesting!)

We do not assume previous knowledge of curves or surfaces; in chapter 2 we start the study of curves from scratch, biassing our account very much in favour of singularities of real-valued functions on the curves and contact with lines, circles, planes and spheres.

The book splits, we hope not literally, into two parts. The first seven chapters are probably enough for a final year (or, in some places, penultimate year) undergraduate course of 30–35 lectures. Chapter 11 could also give an informal finale to such a course. Smooth curves in the plane and in space, contact, envelopes, smooth parametrized manifolds and unfoldings are studied in detail, and many applications and illustrations of these ideas are given both in passing and more systematically in chapter 7. (We kept thinking of more applications, but had to call a halt somewhere.) We hope that a course of this kind might provide a useful antidote to the currently fashionable packaged and axiomatic courses at the undergraduate level, and might also bring back some real geometry to degree courses. (Is it unreasonable to believe that a thorough grasp of something rather concrete and not too difficult can be better than a vague understanding of high-faluting technical mathematics?) The only pre-requisites are some knowledge of linear algebra (matrices and linear maps) and several-variable calculus. (We state and use, but do not prove, the inverse function theorem.) One comparatively deep and difficult result, which gives the existence of 'universal unfoldings', is discussed in some detail, but not proved, in chapter 6.

We regard chapters 8–10 as somewhat more sophisticated than chapters 1–7; they are suitable for relatively advanced undergraduates or first year postgraduates. The pace is rather faster and we assume a good understanding of what has gone before as well as some familiarity with ideas such as compactness. Also in chapter 9 not all the details of all proofs are given. We cover transversality and the application of this crucial idea, via Thom's transversality lemma, to the proofs of results about 'general' curves – properties which hold for 'almost all' curves. In chapter 10 we give a proof of the 'universal unfolding' theorem used in chapter 6, not in the full generality assumed there (such is neither desirable nor feasible in a course at this level), but under the assumption that our functions, curves and so on are *analytic* rather than just smooth. Thus our treatment is complete in this case, which is quite sufficiently interesting in its own right. The proof given in chapter 10 is not the 'standard' one via the preparation theorem, but instead is an adaptation of a more general argument of Kas and Schlessinger. We believe it shows rather clearly why the criterion for a versal unfolding has the form that it does. The first part of chapter 10 is not at all technical. Finally, in chapter 11, we briefly survey the prospect from the vantage point gained so far.

We have included a large number of exercises, many with hints for solution. We believe that mathematics can only be truly said to be going on when the mathematician (professional or student, but we are all students of a kind) is solving problems, and one of the attractions of this subject matter is that it lends itself very well and at every stage to the solution of explicit problems. We also hope that the inclusion of numerous exercises will encourage nonexperts to try teaching this concrete and (we hope) enjoyable material, much of which has not appeared before in book form.

The results of some exercises are used later in the book; these are always provided with adequate hints, so can be regarded as small extensions of the text with some details missing. The sections in small print, on the other hand, are peripheral to the main issues and can be skipped without loss of connexion – though with some loss of entertainment, for there is at least one murky backwater of the subject represented there.

Most of the material in this book has been taught in a final year undergraduate course at Liverpool University or in a first year post-graduate course at University College Cork (or in both). We should like to thank our colleagues at these places, and of course our students, for their help, unwitting or otherwise, in getting the material straight. In particular we thank Professor C. T. C. Wall, of Liverpool, for getting us interested in this subject in the first place. The second author spent much of the writing period as Visiting Professor or Visiting Scholar at the University of North Carolina at Chapel Hill and the University of California at Berkeley. He thanks these institutions for their hospitality, the first of them, and Liverpool University, for financial support, and the

faculty members, particularly Professor J. N. Damon of UNC, for support of a moral and mathematical kind. The computer pictures were produced at Liverpool, using an IBM 4341 computer and a CalComp incremental plotter. The mottoes for chapters and elsewhere all come from the exploits of Sherlock Holmes, in the stories of Sir Arthur Conan Doyle. The frontispiece was drawn for us by Heather Harrison.

Finally we want to thank each other for our patience and forbearance during the writing of this book. We are still friends.

J. W. Bruce
December 1983 P. J. Giblin