1

Introductory example: a gravitational catastrophe machine

'I am afraid that I rather give myself away when I explain,' said he. 'Results without causes are much more impressive.'
(*The Stockbroker's Clerk*)

It is well established that one should never begin a talk – or presumably a book – with an apology. We apologize, therefore, for apologizing that despite the title of this chapter our book is *not* primarily about catastrophe theory. The reason for our beginning with a gravitational catastrophe machine is that it exemplifies, in a vivid way, many of the ideas we shall study in detail later, such as functions on a curve, envelopes, surfaces, projections, evolutes and bifurcation sets. These ideas are merely touched on in the present chapter: do not expect to understand all the details yet.

The gravitational catastrophe machine was invented by T. Poston and is discussed in the well-known book on the subject (Poston and Stewart, 1978). Other introductions to catastrophe theory can be found in Zeeman (1977), Poston and Stewart (1976), Saunders (1980).

Consider a parabola, cut off by a line (perpendicular to the axis say), as in fig. 1.1. Imagine the region enclosed to be a lamina (thin sheet) that is constrained to move in a vertical plane, resting on a horizontal line; we seek the position of stable equilibrium. We do not assume the lamina to be of uniform density; in fact let its centre of gravity be at the point (a, b) referred to axes x and y as shown, relative to which the equation of the parabola is $y = x^2$. (The point (a, b) is often said to be in the *control space*: the values of a and b control the behaviour of the lamina.) Then we are looking for positions that give (local or global) minima of the potential energy of the lamina. The latter is easy to calculate: it is mgh, where m is the mass, g the acceleration due to gravity and h the height of the centre of gravity above some base line, which we take to be the line on which the lamina rests. Thus, if the parabola makes contact with the line at (t, t^2), then h is the distance from (a, b) to the tangent

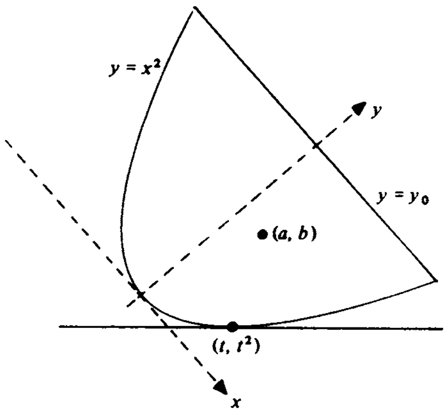

Fig. 1.1. Parabolic lamina in a vertical plane.

$y - 2tx + t^2 = 0$. The potential energy is therefore

1.1 $$V(t) = \frac{b - 2ta + t^2}{(1 + 4t^2)^{\frac{1}{2}}} mg,$$

and the mathematical problem is simply, for given a and b, to find the value or values of t which minimize $V(t)$. We find that

1.2 $$\frac{(1 + 4t^2)^{\frac{3}{2}}}{2mg} V'(t) = U(t), \text{ where } U(t) = 2t^3 + t(1 - 2b) - a,$$

so that we want $U(t) = 0$ for a turning point and U to change from negative to positive at this t for a minimum. Notice that U depends on the given a and b; as a and b *vary* we obtain a *two-parameter family* of functions. Such families will occupy us a good deal later on.

Now $2t^3 + t(1 - 2y) - x = 0$ is precisely the equation of the *normal* to the parabola at (t, t^2). Thus we obtain first the unsurprising fact that for equilibrium (a, b) must be vertically above the point of contact (t, t^2), for the normal at (t, t^2) is the vertical line through that point. We shall return to the question of minima shortly.

It is instructive to regard $2t^3 + t(1 - 2b) - a = 0$ as defining a *surface* M (called the *catastrophe surface* of V) in the three-dimensional space with coordinates (t, a, b). With the t-axis vertical, $t = t_0$ is a *horizontal* plane which meets M in the line $t - t_0 = 2t_0^3 + t_0(1 - 2b) - a = 0$. This is a normal line to the parabola in the plane $t = t_0$ given by the equation $b = a^2$, the normal being at the point $t = t_0$, $a = t_0$, $b = t_0^2$. What this

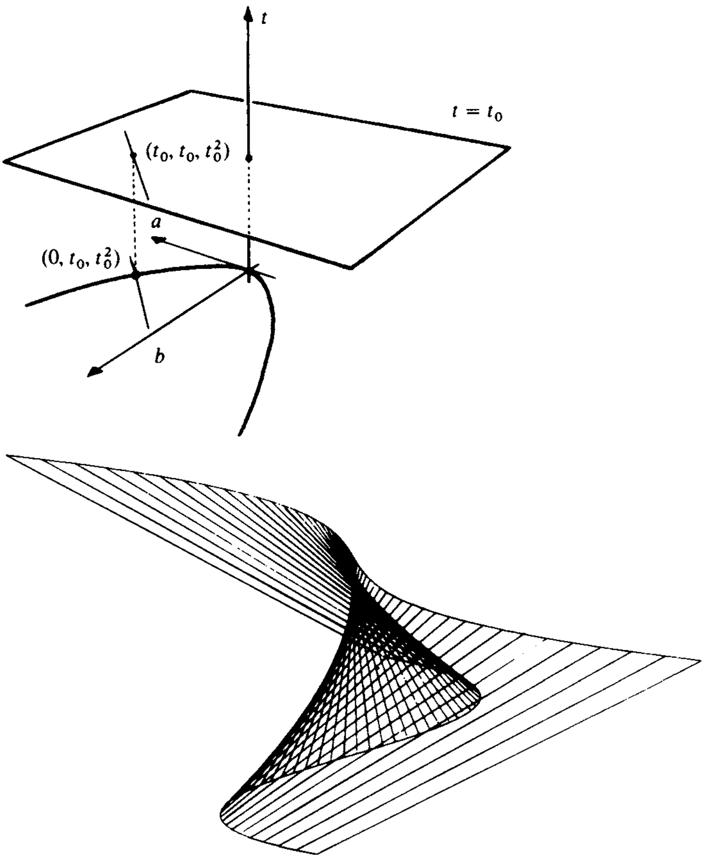

Fig. 1.2. The surface of normals to a parabola.

means geometrically is that M is obtained by taking the normals to the parabola $b = a^2$ in the (a, b)-plane and moving the normal at (t_0, t_0^2) vertically to the height t_0. The normals are spread out to form a surface by using the t-direction* (fig. 1.2).

* It is possible to make a model of the surface M using cotton threads for these lines, passing through holes drilled in perspex (or wooden) sides b = const. Here, in centimetre units, are good dimensions to use. Try the curve $32b = a^2$ with planes $b = 0$, $b = 24$, taking $t = 0, \pm 0.8, \pm 1.6, \ldots, \pm 16$. The face $b = 0$ can be about 36 in t-direction $\times$ 24 in a-direction ($(t, a, b) = (0, 0, 0)$ at the centre) and the face $b = 24$ can be about 36 in t-direction $\times$ 12 in a-direction ($(0, 0, 24)$ at the centre).

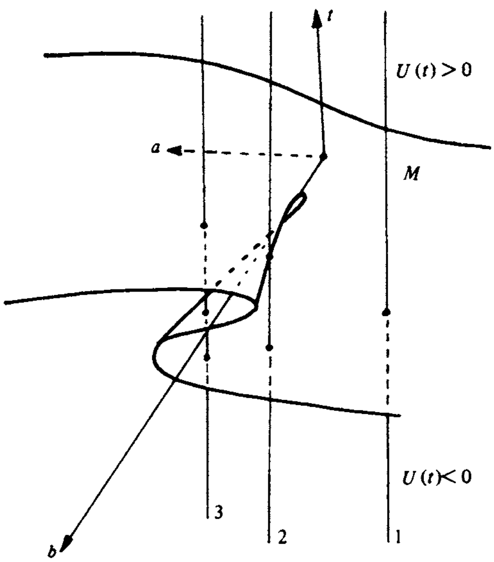

Fig. 1.3. The catastrophe surface.

The *vertical* plane $b = b_0$ meets M in the cubic curve $2t^3 + t(1 - 2b_0) - a = 0$. As b increases through $b_0 = \frac{1}{2}$ this curve acquires a maximum and a minimum, thinking of a as a function of t.

1.3 Exercise

Sketch the cubic curve in the plane $b = b_0$ for $b_0 = 0, \frac{1}{2}$ and $\frac{3}{2}$.

We draw the folded surface M as in fig. 1.3.

For given a and b the vertical line through $(0, a, b)$ meets M in points (t, a, b), which are solutions of $U(t) = 0$ (see 1.2). The number of solutions depends on (a, b); sometimes it is one, sometimes two and sometimes three. The values of t correspond to turning points of the potential energy $V(t)$ (see 1.1), and so to possible stable equilibrium positions for the parabolic lamina.

M divides the (t, a, b)-space into two regions: that which is mostly 'above' M has $U(t) > 0$ and the other has $U(t) < 0$. Minima of V occur for values of t at which $U(t)$ changes from negative to positive. Thus when there is *one* solution to $U(t) = 0$ it is a minimum; when there are *three* the middle one is a maximum and the other two are minima; when there are *two*, so that the vertical line touches M (M has vertical tangent plane),

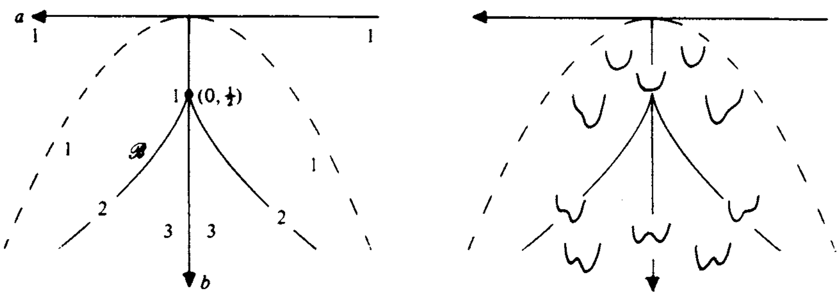

Fig. 1.4. Number of solutions of $U(t) = 0$, and sketches of V for various positions of (a, b).

the point of contact is neither maximum nor minimum ($U(t)$, hence $V'(t)$ remains the same sign) and the other solution is a minimum. The inner part of the fold gives entirely maxima.

Let us find the set $\mathscr{B}$ in the (a, b)-plane consisting of points where there are *two* solutions, i.e. the equation $U(t) = 0$ has a repeated root. The condition for t to be a repeated root is $U(t) = U'(t) = 0$, i.e.

1.4 $$2t^3 + t(1 - 2b) - a = 0$$
$$6t^2 + (1 - 2b) = 0$$

and elimination of t gives

1.5 $$27a^2 = 2(2b - 1)^3.$$

Note that $a = 0$, $b = \frac{1}{2}$ is very special: this gives a triple root, for $U(t) = t^3$ in this case, with triple root 0. The curve $\mathscr{B}$ given by 1.5 is a cuspidal cubic in the (a, b)-plane, i.e. in the control space (fig. 1.4).

1.6 Exercises

(1) Show that, if t is the repeated root of $U(t) = 0$, where $(a, b) \in \mathscr{B}$, then $a = -4t^3$ and $b = \frac{1}{2}(1 + 6t^2)$. This gives a *parametrization* of $\mathscr{B}$ (with parameter t). The points (t, a, b) of M for which $(a, b) \in \mathscr{B}$ therefore have the form $(t, -4t^3, \frac{1}{2}(1 + 6t^2))$.
This is the curve in M which projects to $\mathscr{B}$ in the (a, b)-plane. It is a *space curve* and it has the *regular parametrization* given (this means that the derivatives of the three component functions never vanish for the same t – indeed here the derivative of the first component never vanishes at all). Is the tangent line to the space curve ever vertical?

(2) Sketch the potential function V for various values of (a, b), such as $(0, 0)$, $(0, 1)$, $(1, \frac{1}{2})$. Compare with the rough sketches in fig. 1.4. You may find

it helpful to use

$$V'(t) = 2(1 + 4t^2)^{-\frac{3}{2}}(2t^3 + t(1 - 2b) - a)$$

and

$$V''(t) = 2(1 + 4t^2)^{-\frac{5}{2}}((16b - 2)t^2 + 12at + 1 - 2b).$$

The set $\mathscr{B}$, which is variously called the *bifurcation set* of V or the *discriminant set* of U, separates (a, b) giving one solution to $U(t) = 0$ ('outside' $\mathscr{B}$, $>$ in 1.5) from those giving three solutions ('inside' $\mathscr{B}$, $<$ in 1.5). Looking at the surface M from above, a curve like $\mathscr{B}$ is what we 'see' of the folded surface: it is the 'apparent contour' where the surface folds away from us (likewise the apparent contour of a sphere held at arm's length is a circle).

There is another way of regarding this. The first equation of 1.4 is the equation of the normal to the parabola $b = a^2$ at (t, t^2): as t varies but a and b remain fixed, it gives the family of all normals to the parabola. Eliminating t between the equations of 1.4 amounts to finding the *envelope* of this family of lines: a curve which is touched by each of the lines of the family (see 1.7 below). The envelope of the normals to a curve is also called the *evolute* of the curve, so $\mathscr{B}$ is the evolute of the parabola $b = a^2$ (fig. 1.5).

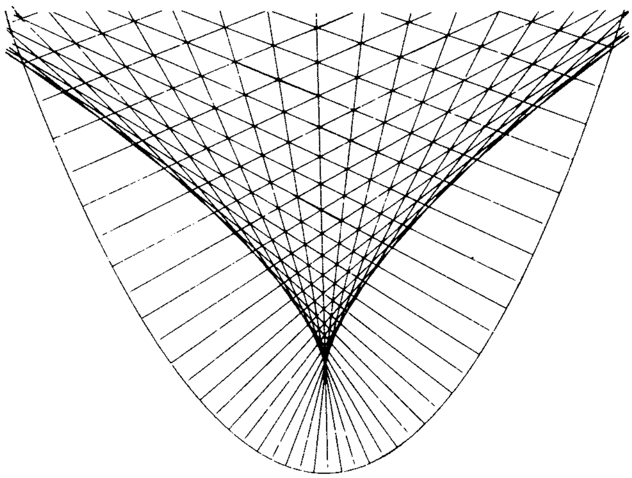

Fig. 1.5. Normals to a parabola and their envelope.

1.7 Exercise
Show that the curve 1.5 consists precisely of points of the form $(-4t^3, \frac{1}{2}(1+6t^2))$ for $t \in \mathbb{R}$. Show that the tangent to 1.5 at the point given by any value of t other than 0 coincides with the normal to the parabola $b = a^2$ at the point (t, t^2). The missing normal, at $(0, 0)$, is the line $a = 0$. Is this in some sense the 'tangent' to 1.5 at $(0, \frac{1}{2})$?

We could go on conjuring more geometry from the parabolic lamina, but let us return to finding stable equilibria. For a given position (a, b) of the centre of mass in the control space we now know (fig. 1.4) how many solutions there are of $U(t) = 0$, i.e. how many stationary points there are of the potential V, and also which are minima. Naturally for a physical lamina the centre of gravity will be at some point of the lamina itself, i.e. (a, b) will satisfy $a \leqslant b^2$. The exact values of t can be found by solving the cubic equation $2t^3 + t(1-2b) - a = 0$.

It is of particular interest to see what happens to the stable equilibria if we *move* the centre of gravity (a, b) around in control space. We are not concerned here with the dynamics of rolling, merely with the change in the positions of stable equilibrium.

Suppose for example that the point (a, b) moves steadily across $\mathscr{B}$, along the line 12345 in fig. 1.6. Corresponding to 1 there is only *one* stable equilibrium, given by the point 1′ of M. At 2 and 3 there are two

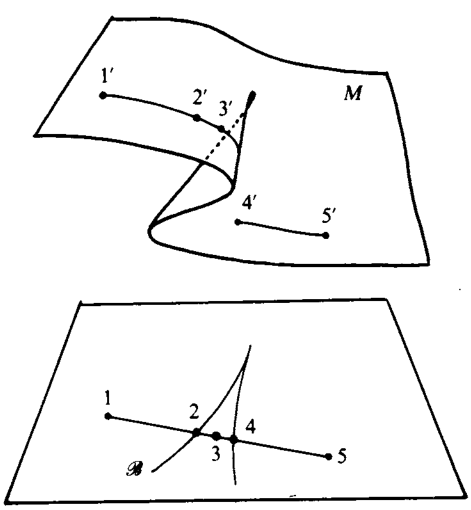

Fig. 1.6. Catastrophe surface and its projection to control space.

Fig. 1.7. Catastrophic change of equilibrium.

equilibrium positions available, but in practice the parabola tilts slowly so that positions given by 2′ and 3′ are taken. As (a, b) moves past 4 (which is on $\mathscr{B}$), however, no continuous change in the equilibrium position is possible: the path on M above 345 has a break in it at 4′, where the parabola suddenly spins to the left, a large change in equilibrium position occurring from a small shift in (a, b). This is called a *catastrophe* (fig. 1.7). It is well worth making a model to illustrate this: as described in Poston and Stewart (1978), one can be made from two parallel parabolas made out of thin card, held apart by three or four polystyrene 'pillars'. A magnet and disc of metal on opposite sides of one card can be moved about to change (a, b). Because of the weight of the card, the positions of the *magnet* at which catastrophes occur will not coincide exactly with the bifurcation set $\mathscr{B}$, but can be determined by experiment.

1.8 Exercises

(1) Describe the behaviour of the lamina as (a, b) moves (i) along the path 54321 in fig. 1.6; (ii) clockwise round a circle centred at the cusp in fig. 1.6; (iii) anticlockwise round the same path. In each case sketch the change in the potential function V as (a, b) traverses the path.

(2) How many normals to the parabola with equation $y = x^2$ pass through a point (a, b) inside the parabola?

(3) A swimmer gets into difficulties in a parabolic cove, and needs to head for the nearest point of land. How many choices does he have? This is similar to, but not identical with, the above example. We should expect him to choose the *absolute* minimum distance always, and the axis of the parabola is now significant. How does the nearest point of shore change as the swimmer's initial position moves across the cove? Does it move continuously, or is there a catastrophe? The reader is recommended *not* to try a practical experiment for this exercise. (This example was suggested to us by Dr I. R. Porteous.)

One final point. Consider the potential function V corresponding to the *cusp point* $(a, b) = (0, \frac{1}{2})$, namely

$$V(t) = \frac{1 + 2t^2}{2(1 + 4t^2)^{\frac{1}{2}}}.$$

This has a *degenerate minimum* at $t = 0$, that is $V'(0) = V''(0) = V'''(0) = 0$ but $V^{(4)}(0) \neq 0$. Functions which have degenerate minima are to be regarded as *exceptional*: most functions do not have them. However, we *do* expect to find an occasional function with a degenerate minimum amongst a two-parameter family of functions such as the potential functions V which depend on a and b. The occasional member of a general family will be worse behaved than is a general function, and the bigger the family, the worse the behaviour we can expect. This rather ominous maxim will be made a little more precise in 6.20.

1.9 Project

An even more interesting catastrophe machine was invented by E. C. Zeeman, and is described in various places, such as Zeeman (1977). Find out about Zeeman's machine, its potential function, and the sudden changes which occur as the control parameters (corresponding to a, b above) are varied. (Why not build one too?)

2

Curves, and functions on them

'I think, Watson, that you have put on seven and a half pounds since I last saw you.'
'Seven,' I answered.
'Indeed, I should have thought a little more. Just a trifle more, I fancy, Watson.'
(*A Scandal in Bohemia*)

Plane curves arise naturally in all sorts of situations and in many guises. Solutions of Newton's laws of motion give the orbits of the planets as ellipses with the Sun at a focus. A spot of paint on a train wheel describes a cycloid as the wheel rolls. These are examples of curves parametrized by time: for each time t a definite point on the curve is determined. If a solid object (such as Dr Watson) is viewed from a distance its outline, also called its apparent contour or profile, is essentially a plane curve (or a curve on the retina), but this time it is not given dynamically as a moving point (fig. 2.1). It is more reminiscent of curves given by equations $f(x, y) = 0$; these latter curves are one of the subjects of chapter 4. A curve may be traced by a linkage of bars and gearwheels; the position of the pencil drawing the curve perhaps depends on the angle of some controlling bar, and so is parametrized by this angle. (Alas! We have no space for this beautiful subject.) When the Sun's rays are reflected from the rounded inner surface of a teacup they produce on the surface of the tea a bright 'caustic' curve. The reflected rays are all tangents to this curve, which is said to be the 'envelope' of the rays. We study envelopes in chapter 5 and caustics at the end of chapter 7.

We are interested also in space curves (curves in $\mathbb{R}^3$) and to a limited extent in curves in $\mathbb{R}^n$. For the present we look at parametrized curves and coax the geometry from them by means of real-valued functions defined on the curves. To see how this can be geometrically interesting and to set the scene here is an example.