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113 Spectral Decomposition and Eisenstein Series

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Spectral Decomposition and Eisenstein Series

Une Paraphrase de l'Écriture

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Preamble

Over the last few years, Arthur has formulated some extremely precise conjectures generalising Jacquet's conjectures for $GL(n)$, on the description of square integrable automorphic forms (see [J], [A2], [A3], [A4]). The origin of these conjectures can be found in his work on the trace formula and their point of departure goes back to Langlands' work (see Arthur's Corvallis talk [A1]). It is probable that the full force of the constructions of these automorphic forms as residues of Eisenstein series has not been exploited. It is thus important to fully understand the basic book [L]; this was our main motivation. Since Langlands wrote the material of [L] (around 1967), several authors have already given personal presentations (Godement [G], Harish-Chandra [HC], Osborne-Warner [OW]). Morris extended Langlands' results to the function field case ([M1] and [M2]).

The following notes are a reworking of the book [L] and an ameliorated (and unified) version of talks which we gave in the 'automorphic' seminar (Paris 7/ENS). A seminar cannot exist without critical auditors: we wish to thank P. Barrat, J.-P. Labesse, P. Perrin (who also gave some talks on this subject), A.-M. Aubert, C. Blondel, L. Clozel, G. Henniart, G. Laumon (for whom the goal of the seminar was to render entirely obscure what was already not particularly clear), M.-F. Vignéras and D. Wigner. We also thank Y. Colin de Verdières who gave a talk on his proof of the meromorphic continuation of Eisenstein series [C], K. Lai for his remarks on a first version of the manuscript, P. Deligne who communicated to us the proof of Appendix I and finally, most particularly, H. Jacquet who explained to us the proof of the meromorphic continuation of Eisenstein series given in Chapter IV.

Here is a more precise description of the different chapters.

The first chapter is a general discussion of automorphic forms and their constant terms. Our goal is not to give a complete description of their properties, but only to formulate these properties in the framework which we need, with particular emphasis on those properties resulting from the principle according to which an automorphic form is determined by its constant terms. In particular, we use without proof some results of reduction theory and of the theory of cuspidal automorphic forms. We are mainly inspired by Harish-Chandra [HC], Godement [G], Borel and Jacquet [BJ] and of course Langlands [L], Chapters 1, 2, 3 and 5.

Let k be a global field, G a connected reductive group defined over k, $G(\mathbb{A})$ the group of its adelic points. We take a group $\mathbf{G}$ which is a finite central covering of $G(\mathbb{A})$ in which the group $G(k)$ of rational points lifts to a group also denoted by $G(k)$. This framework allows us to include the metaplectic groups. After a long paragraph devoted to the properties of these objects, we introduce the notion of constant term of a function on $G(k)\backslash\mathbf{G}$: let $P = MU$ be a standard parabolic subgroup of G and let ϕ be a function, let us say continuous, on $G(k)\backslash\mathbf{G}$; its constant term along P is the function ϕ_P on $U(\mathbb{A})M(k)\backslash\mathbf{G}$ defined by

$$\phi_P(g) = \int_{U(k)\backslash U(\mathbb{A})} \phi(ug)\mathrm{d}u,$$

see I.2.6. We explain how a function with moderate growth can be approximated by a linear combination of its constant terms along the different standard parabolic subgroups of G (see I.2.7 to I.2.16; the paragraphs I.2.13 to I.2.16 come from [A2]). We then introduce the notions of automorphic form and cuspidal automorphic form on $G(k)\backslash\mathbf{G}$, or more generally on $U(\mathbb{A})M(k)\backslash\mathbf{G}$, $P = MU$ being a standard parabolic subgroup, see I.2.17, I.2.18.

Suppose for a moment that the center $Z_{\mathbf{G}}$ of $\mathbf{G}$ is trivial, and let ϕ be an automorphic form on $G(k)\backslash\mathbf{G}$. We easily define the 'cuspidal component' of ϕ: it is the unique cuspidal automorphic form ϕ^{cusp} on $G(k)\backslash\mathbf{G}$ such that

$$\langle\phi_0, \phi^{\text{cusp}}\rangle = \langle\phi_0, \phi\rangle$$

for every cuspidal automorphic form ϕ_0, where we have set

$$\langle\phi_0, \phi\rangle = \int_{G(k)\backslash\mathbf{G}} \overline{\phi}_0(g)\phi(g)\mathrm{d}g.$$

This definition extends to the case of an arbitrary group $\mathbf{G}$ and an automorphic form on $U(\mathbb{A})M(k)\backslash\mathbf{G}$, see I.2.18 and [L], p. 55. In particular, if ϕ is an automorphic form on $G(k)\backslash\mathbf{G}$, we define the cuspidal component ϕ_P^{cusp} of the constant term of ϕ along P. When P runs through the standard parabolic subgroups, these functions determine ϕ (Proposition I.3.4, [L] Lemma 3.7).

Let $P = MU$ be a standard parabolic subgroup and π an irreducible cuspidal automorphic representation of $\mathbf{M}$ (:= the inverse image of $M(\mathbb{A}) \in \mathbf{G}$). We define the space $A_0\big(U(\mathbb{A})M(k)\backslash\mathbf{G}\big)_\pi$ of cuspidal forms of type π on $U(\mathbb{A})M(k)\backslash\mathbf{G}$, see I.3.3.

Denote by $\mathrm{Rat}(M)$ the group of rational characters of M,

$$M(\mathbb{A})^1 := \bigcap_{\chi\in\mathrm{Rat}(M)} \mathrm{Ker}\,|\chi|$$

and by $\mathbf{M}^1$ its inverse image in $\mathbf{G}$. The abelian group $\mathbf{M}/\mathbf{M}^1$ is a real finite-dimensional vector space if k is a number field, a finitely-generated free $\mathbb{Z}$-module if k is a function field. This allows us to define the space of polynomials on $\mathbf{M}/\mathbf{M}^1$, which we have reasons to denote by $\mathbb{C}[\mathrm{Re}\,\mathfrak{a}_M]$. Such a polynomial defines a function on $\mathbf{G}$, left invariant under $\mathbf{M}^1U(\mathbb{A})$.

This being so, for π as above,

$$(1) \qquad \mathbb{C}[\mathrm{Re}\,\mathfrak{a}_M] \otimes_{\mathbb{C}} A_0\big(U(\mathbb{A})M(k)\backslash\mathbf{G}\big)_\pi$$

can be identified with a subspace of the space of cuspidal automorphic forms on $U(\mathbb{A})M(k)\backslash\mathbf{G}$. This last is in fact the sum of these subspaces, when π runs through all the irreducible cuspidal representations of $\mathbf{M}$, see I.3.3.

This allows us to define the support of a cuspidal form ϕ as the smallest set Π of irreducible cuspidal representations of $\mathbf{M}$ such that ϕ belongs to the sum of the spaces (1) for $\pi \in \Pi$. In particular, if ϕ is automorphic on $G(k)\backslash\mathbf{G}$, ϕ_P^{cusp} has a support which we denote by $\Pi_0(\mathbf{M}, \phi)$.

Knowledge of this support, now as P runs through the standard parabolic subgroups, determines upper bounds of the function ϕ (see Lemma I.4.1). Using arguments of [L], Lemmas 5.1 and 5.2, and an argument of Waldschmidt (see Lemma I.4.2), we obtain relatively precise criteria for the convergence of a sequence of automorphic forms (Lemma I.4.4) or for a function $z \mapsto \phi_z$ with values in the space of automorphic forms (Lemma I.4.10) to be holomorphic. These results are somewhat stronger than those in Chapter 5 of [L]. Finally, if we suppose that ϕ has a unitary central character, the fact that ϕ is or is not square integrable (modulo Z_G) can be read off the central characters of the elements of its supports $\Pi_0(\mathbf{M}, \phi)$ (see I.4.11, [L] pp. 104 and 186).

In Chapter II, we introduce the basic objects, Eisenstein series and what we call pseudo-Eisenstein seris (Godement calls them theta series and Langlands does not really call them by any precise name); they are in fact integrals of Eisenstein series: their properties which we use are proved in Chapter 4 of [L]. These objects have become classical and we were largely inspired by Godement's Bourbaki talk presenting Langlands' work. They depend first and foremost on the given standard Levi subgroup M of G whose inverse image in $\mathbf{G}$ we denote by $\mathbf{M}$. We denote by $X_M^{\mathbf{G}}$ the set of (continuous) characters of $\mathbf{M}$ trivial on $\mathbf{M}^1$ (see above) and on the centre of $\mathbf{G}$ (a precise description of this group is given in I.3; if k is a number field, $X_M^{\mathbf{G}}$ is naturally a vector subspace of Rat $M \otimes_{\mathbb{Z}} \mathbb{C}$). They also depend on an orbit of $X_M^{\mathbf{G}}$ in the set of (not indispensably) irreducible automorphic subrepresentations of $\mathbf{M}$, denoted by $\mathfrak{P}$, i.e. $\mathfrak{P}$ is of the form:

$$\mathfrak{P} := \{\lambda \otimes \pi_0,\ \lambda \in X_M^{\mathbf{G}}\},$$

where π_0 is a representation of $\mathbf{M}$ into the space of automorphic forms on $\mathbf{M}$. Finally, they depend on sections of the fibre bundle over $\mathfrak{P}$ whose fibre over a point π of $\mathfrak{P}$ is the space of automorphic forms on $M(k)U(\mathbb{A})\backslash\mathbf{G}$ of type π (see II.1.1) (where U is the unipotent radical of the standard parabolic of Levi M).

Note that if k is a number field, it is possible to fix π_0 canonically (requiring it to have a central character which is trivial on a subtorus (in the sense of Lie groups) of $\mathbf{M}$ which is a supplementary of $Z_{\mathbf{G}}\mathbf{M}^1$). Thus, in this case, we obtain a canonical identification of $X_M^{\mathbf{G}}$ with $\mathfrak{P}$, and the fibre bundle which interests us is canonically isomorphic to the trivial fibre bundle on $X_M^{\mathbf{G}}$ whose fibre is the space of automorphic forms on $M(k)U(\mathbb{A})\backslash\mathbf{G}$ of type π_0. On the other hand, if k is a function field, there is in general no canonical choice of π_0 and the fibre bundle which interests us is isomorphic to a principal fibre bundle over $X_M^{\mathbf{G}}/\mathrm{Fix}_{X_M^{\mathbf{G}}}\mathfrak{P}$ of fibre the space of automorphic forms on $M(k)U(\mathbb{A})\backslash\mathbf{G}$ of type π_0.

It is not difficult to equip $X_M^{\mathbf{G}}$ and $\mathfrak{P}$ with structures of complex analytic manifolds such that for any choice of $\pi_0 \in \mathfrak{P}$ the map:

$$\lambda \in X_M^{\mathbf{G}} \mapsto \lambda \otimes \pi_0 \in \mathfrak{P}$$

is holomorphic. We then easily define the notion of holomorphic, meromorphic, Paley–Wiener, etc. sections of the above fibre bundle. Let ϕ be a section: we classically define an Eisenstein series depending on $\mathfrak{P}$ by setting, when the series converges absolutely,

$$E(\phi, \pi)(g) = \sum_{\gamma' in P(k)\backslash G(k)} \phi(\pi)(\gamma g);$$

closely following Godement, we give the classical sufficient conditions which ensure the absolute and uniform convergence on all compact subsets of $\mathbf{G}$, conditions on the absolute value of the central character of π, denoted by $\operatorname{Re}\pi$.

In what follows, we fix a unitary character of the centre of $\mathbf{G}$, denoted by ξ, and we suppose that $\mathfrak{P}$ consists of cuspidal representations of $\mathbf{M}$, the restriction of whose central character to the centre of $\mathbf{G}$ is ξ, and we consider only the sections ϕ as above with values in the space of cuspidal automorphic forms on $M(k)U(\mathbb{A})\backslash\mathbf{G}$. We then say that $(M, \mathfrak{P})$ is a cuspidal datum relative to ξ. For $\mathfrak{P}$ satisfying these hypotheses and for $\pi \in \mathfrak{P}$, the character $\operatorname{Re}\pi$ mentioned above is in fact an element of $X_M^{\mathbf{G}}$. We can define the pseudo-Eisenstein series for $(M, \mathfrak{P})$ fixed as above and for ϕ a Paley-Wiener section, by setting:

$$\theta_\phi := \int_{\substack{\pi\in\mathfrak{P} \\ \operatorname{Re}\,\pi=\lambda_0}} E(\phi, \pi)\mathrm{d}\pi,$$

where λ_0 is a very positive real element of $X_M^{\mathbf{G}}$. We check that θ_ϕ is well-defined (i.e. that convergence of the integral does not depend on the choice of λ_0 and that the integral is a rapidly decreasing function on $G(k)\backslash\mathbf{G}$ (with compact support if k is a function field). One of the properties of pseudo-Eisenstein series is the following 'density' theorem (see II.1.12):

Let ψ be a function on $G(k)\backslash\mathbf{G}$ on which the center of $\mathbf{G}$ acts via ξ, and which is either slowly increasing or L^2 modulo the centre, such that:

$$\int_{Z_{\mathbf{G}}G(k)\backslash\mathbf{G}} \overline{\psi}(g)\theta_\phi(g)\mathrm{d}g = 0,$$

for every pseudo-Eisenstein series (we obviously allow any cuspidal datum relative to ξ). Then $\psi = 0$.

This result is in fact a more or less immediate consequence of Chapter I. The second part of Chapter II is devoted to the calculation of the scalar product of two pseudo-Eisenstein series. This calculation is standard: it is done by calculating the constant terms of either Eisenstein series ([L], 4.6(ii)) or of pseudo-Eisenstein series. It is the second point of view which we adopt (II.2.1). The calculation of the constant terms of Eisenstein series is done in II.1.7 and that of the constant terms of pseudo-Eisenstein series in II.2.2; these calculations are similar and are done in a relatively general framework, already used by Arthur. The calculation of the scalar product gives in particular the first orthogonal decomposition of $L^2(G(k)\backslash\mathbf{G})_\xi$ (see [L], 4.6(i)):

Let $(M, \mathfrak{P})$ and $(M', \mathfrak{P}')$ be cuspidal data relative to ξ; we say that $(M, \mathfrak{P})$ is equivalent to $(M', \mathfrak{P}')$ if there exists $\gamma \in G(k)$ such that $\gamma M\gamma^{-1} = M'$ and $\gamma\mathfrak{P} = \mathfrak{P}'$. Let $\mathfrak{X}$ be an equivalence class of cuspidal data: we denote by

$L^2(G(k)\backslash\mathbf{G})_{\mathfrak{X}}$ the closure of the subspace of $L^2(G(k)\backslash\mathbf{G})_\xi$ generated by the pseudo-Eisenstein series θ_ϕ corresponding to all the elements of $\mathfrak{X}$. Then if $\mathfrak{X}$, $\mathfrak{X}'$ are two distinct equivalence classes, $L^2(G(k)\backslash\mathfrak{G})_{\mathfrak{X}}$ is orthogonal to $L^2(G(k)\backslash\mathfrak{G})_{\mathfrak{X}'}$ and $L^2(G(k)\backslash\mathbf{G})_\xi$ is the completion of the orthogonal sum $\oplus_{\mathfrak{X}} L^2(G(k)\backslash\mathfrak{G})_{\mathfrak{X}}$ where $\mathfrak{X}$ runs through the set of equivalence classes of cuspidal data.

In fact, for what follows and essentially to solve convergence problems in the case where k is a number field, Langlands remarked that it is too restrictive to work only with pseudo-Eisenstein series coming from Paley-Wiener sections ϕ. Following Langlands, we introduce the notions of R-Paley-Wiener (see II.1.4), and for R large enough, we define θ_ϕ as above. We then show that θ_ϕ is still square integrable modulo the centre (II.1.10): there are no major obstacles to calculating the constant terms of θ_ϕ under this hypothesis or to extending the scalar product formula to this situation. Thanks to the density theorem, we immediately see that $\theta_\phi \in L^2(G(k)\backslash\mathbf{G})_{\mathfrak{X}}$ if ϕ is R-Paley-Wiener and is defined from a cuspidal datum belonging to $\mathfrak{X}$.

Chapter III introduces an algebra of operators. Let $\mathfrak{X}$ be an equivalence class of cuspidal data. For $(M,\mathfrak{P}) \in \mathfrak{X}$, let $f_{M,\mathfrak{P}}$ be a function on $\mathfrak{P}$, say with values in $\mathbb{C}$, to simplify. If certain properties of growth, of regularity and of invariance are satisfied, the family $f = (f_{M,\mathfrak{P}})_{(M,\mathfrak{P})\in\mathfrak{X}}$ determines an operator $\Delta(f)$ on $L^2(G(k)\backslash\mathbf{G})_{\mathfrak{X}}$. In particular, if $(M,\mathfrak{P}) \in \mathfrak{X}$ and ϕ is a Paley-Wiener function on $\mathfrak{P}$, we have $\Delta(f)\theta_\phi = \theta_{\phi'}$, where $\phi' = f_{M,\mathfrak{P}}\phi$. The space $H^R_{\mathfrak{X}}$ of these operators appears as a global analogue of the centre of the enveloping algebra of $\mathfrak{G}_\infty$. In particular, it contains an auto-adjoint operator which plays an essential role in Chapter IV to eliminate questions about convergence at infinity and whose introduction by Langlands is one of his subtlest contributions.

These operators allow us to decompose the space of automorphic forms in the following way. Let h be a conjugacy class of pairs (M,π) where M is a standard Levi and π an irreducible cuspidal representation of $\mathbf{M}$. Let us denote by $A(G(k)\backslash\mathbf{G})_\mathfrak{y}$ the space of automorphic forms ϕ on $G(k)\backslash\mathbf{G}$ such that the set consisting of pairs (M,π), where M is a standard Levi and $\pi \in |Pi_0(\mathbf{M},\phi)$ is contained in $\mathfrak{y}$. Then when we let $\mathfrak{y}$ vary, the spaces $A(G(k)\backslash\mathbf{G})_\mathfrak{y}$ generate the space of automorphic forms on $G(k)\backslash\mathbf{G}$ (see III.2.6).

The chapter ends with a statement reformulating Lemma 7.3 of [L], according to which if ϕ is an automorphic form on $G(k)\backslash\mathbf{G}$, with unitary central character and square integrable modulo $Z_\mathbf{G}$, the central characters χ_π of the elements π of the cuspidal supports $\Pi_0(\mathbf{M},\phi)$ are not arbitrary. To state the result simply, suppose that $\mathbf{G} = G(\mathbb{A})$ and $Z_\mathbf{G} = \{1\}$. Denote by T_M the largest split torus contained in the centre of M. We show that there exists an integer $N(G)$, depending only on G as indicated by the notation, such that for π as above, the restriction to $T_M(\mathbb{A})$ of $(\chi_\pi)^{N(G)}$ has positive real values.

In Chapter IV we prove that the Eisenstein series coming from a pair $(M, \mathfrak{P})$ consisting of a standard Levi and an orbit of irreducible cuspidal automorphic representations of $\mathbf{M}$ can be meromorphically continued to all of $\mathfrak{P}$. This is also true for the intertwining operators. The proof was communicated to us by Jacquet. He actually attributes it to Selberg. The idea of this proof can also be found in [E].

We start with the case where M is a proper maximal Levi of G. The principle of the proof can then be applied, via induction, to all of M. Suppose thus that M is proper and maximal, and suppose to simplify things that k is a number field and $\mathrm{Stab}(M, \mathfrak{P})$ consists of two elements 1 and w. We first show that the truncated Eisenstein series $\wedge^T E(\phi, \pi)$, defined in a positive half-plane, are solutions of functional equations in which only compact operators occur (Lemma IV.3.4). The usual theory of resolvents of compact operators then allows us to construct a function $\tilde{E}$ which is meromorphic on (nearly) all of $\mathfrak{P}$, such that in a positive half-plane, we have the equality:

$$(**) \qquad E(\phi, \pi) = \tilde{E}(\phi, \pi) + \tilde{E}(M(w, \pi)\phi, w\pi),$$

where $M(w, \pi)$ is the intertwining operator. But E itself satisfies a functional equation which $\tilde{E}$ does not satisfy. Applying this equation, we deduce from (**) the equality

$$0 = R(\phi, \pi) + R(M(w, \pi)\phi, w\pi),$$

see IV.3.9. From this, we deduce by inversion the meromorphic continuation of $M(w, \pi)$. We then continue $E(\phi, \pi)$ by the equality (**).

Using the lemmas of I.4 and the operators of Chapter III, we prove diverse properties of the singularities of Eisenstein series and intertwining operators. We also obtain the functional equations which in the above situation can be written

$$E(\phi, \pi) = E(M(w, \pi), w\pi)$$

$$M(w, w\pi)M(w, \pi) = 1.$$

Finally, we reproduce an argument of Harder ([H] I.6.6) which shows that, if k is a function field, the operators $M(w, \pi)$ and the values $E(\phi, \pi)(g)$ (for fixed g and suitable ϕ) are rational functions of π.

The objective of Chapters V and VI is to give the 'spectral' decomposition of $L^2(G(k)\backslash\mathbf{G})_{\mathfrak{X}}$ where $\mathfrak{X}$ is an equivalence class of cuspidal data relative to the character ξ (see above). For these chapters, we follow practically word for word Chapter VII and Appendix II of Langlands. Thus we fix $\mathfrak{X}$ and for $(M, \mathfrak{P}) \in \mathfrak{X}$, we define $S_{(M,\mathfrak{P})}$ (see V.1.1) to be a set of 'affine subspaces' of $\mathfrak{P}$ which contains all the singular hyperplanes of the intertwining operators:

$$\pi \mapsto M(w^{-1}, -w\overline{\pi})\phi(-w\overline{\pi}),$$

where w is an element of the Weyl group of G of minimal length in its right coset modulo the Weyl group of M and such that wMw^{-1} is still a standard Levi of G (we denote this set by $W(M)$; it is evidently not a group), $-\overline{\pi}$ is the Hermitian contragredient of π and ϕ is a holomorphic section of the fibre bundle over $-w\overline{\mathfrak{P}}$ which is the obvious analogue of the one described above.

We suppose moreover that $S_{(M,\mathfrak{P})}$ is stable under intersection and under conjugation in the following sense:

$$\forall w \in W(M),\ wS_{(M,\mathfrak{P})} = S_{(wMw^{-1},w\mathfrak{P})}.$$

We set $S_{\mathfrak{X}} = \cup_{(M,\mathfrak{P})\in\mathfrak{X}} S_{(M,\mathfrak{P})}$. Let $\mathfrak{G}', \mathfrak{G}'' \in S_{\mathfrak{X}}$; we say that $\mathfrak{G}'$ and $\mathfrak{G}''$ are equivalent if $\mathfrak{G}' \in S_{(M',\mathfrak{P}')}$, $\mathfrak{G}'' \in S_{(M'',\mathfrak{P}'')}$ with $(M', \mathfrak{P}'), (M'', \mathfrak{P}'') \in \mathfrak{X}$ and if there exists $w \in W(M')$ such that $w\mathfrak{G}' = \mathfrak{G}''$ (which also implies that $wM'w^{-1} = M''$ and $w\mathfrak{P}' = \mathfrak{P}''$). We denote by $[S_{\mathfrak{X}}]$ the set of equivalence classes. We note that $S_{\mathfrak{X}}$ is a locally countable set and that it becomes locally finite if we fix a finite number of $\mathbf{K}$-types $\mathfrak{F}$, and we require that the action on ϕ above be via one of these $\mathbf{K}$-types. This is the point of view of Langlands, which we have also adopted, following him, but we do not develop it in this introduction in order to keep the notation reasonable. The goal is to associate with every element $\mathfrak{C}$ of $[S_{\mathfrak{X}}]$ a closed subspace of $L^2(G(k)\backslash\mathbf{G})_{\mathfrak{X}}$, denoted by $L^2(G(k)\backslash\mathbf{G})_{\mathfrak{C}}$ and possibly zero, such that

(1) $L^2(G(k)\backslash\mathbf{G})_{\mathfrak{C}}$ is orthogonal to $L^2(G(k)\backslash\mathbf{G})_{\mathfrak{C}'}$ if $\mathfrak{C}$ is different from $\mathfrak{C}'$,

(2) $L^2(G(k)\backslash\mathbf{G})_{\mathfrak{X}}$ is the completion of the orthogonal sum of the

$$L^2(G(k)\backslash\mathbf{G})_{\mathfrak{C}},$$

the discrete spectrum of $L^2(G(k)\backslash\mathbf{G})_{\mathfrak{X}}$ is the completion of the orthogonal sum of $L^2(G(k)\backslash\mathbf{G})_{\mathfrak{C}}$ when $\mathfrak{C}$ consits of elements of dimension 0, i.e. points.

These results from [L] do not have the abstract form given here, but [L] gives at the same time a very explicit method for constructing the orthogonal projection of $L^2(G(k)\backslash\mathbf{G})_{\mathfrak{X}}$ onto $L^2(G(k)\backslash\mathbf{G})_{\mathfrak{C}}$. We denote this projection by $\text{proj}_{\mathfrak{C}}$. For this, we describe $\text{proj}_{\mathfrak{C}}\,\theta_\phi$ when ϕ is R-Paley-Wiener. This projection is an integral of residues of Eisenstein series. Thus there are several problems to solve:

(a) These residues must be defined: we do this with, as our only tool, the classical residue theorem generalised to several variables. For us the definitions and the residue theorem form the object of V.1.5 and the description of the residues is in V.2.2. These residues are denoted by $\mathrm{Res}^G_{\mathfrak{S}'} E(\phi, \pi)$ for $\mathfrak{S}' \in \mathfrak{C}$ and $\pi \in \mathfrak{S}'$. (These references correspond to [L], 7.1, which is more precise than V.1.5 as we explain there, and to [L], proof of 7.7).

(b) The set over which we integrate must be defined: for this we fix $\mathfrak{S} \in \mathfrak{C}$ (everything that follows is independent of this choice) and we integrate over the imaginary axis of $\mathfrak{S}$ defined as follows: we recall that $\mathrm{Re}\,\mathfrak{S} := \{\mathrm{Re}\,\pi, \pi \in \mathfrak{S}\}$ is an affine subspace (in the usual sense) of $\mathrm{Rat}\, M \otimes_{\mathbf{Z}} \mathbb{R}$; we write $(\mathrm{Re}\,\mathfrak{S})^0$ for its vector part and we set:

$$o(\mathfrak{S}) = (\mathrm{Re}\,\mathfrak{S})^{0\perp} \cap \mathrm{Re}\,\mathfrak{S},$$

where the orthogonal is taken for an invariant scalar product with which we equip $\mathrm{Rat}\, M \otimes_{\mathbf{Z}} \mathbb{R}$. The imaginary axis is then:

$$\{\pi \in \mathfrak{S} \mid \mathrm{Re}\,\pi = o(\mathfrak{S})\}.$$

(c) The function to be integrated must be defined: we must show that it is holomorphic at every point of the set of integration and that the integral converges in an appropriate sense. For the function, let $\mathfrak{S}$ be as above and $(M, \mathfrak{P}) \in \mathfrak{C}$ such that $\mathfrak{S} \in S_{(M,\mathfrak{P})}$: we take

$$\{\pi \in \mathfrak{S} \mid \mathrm{Re}\,\pi = o(\mathfrak{S})\} \mapsto \sum_{w \in W(M)} \left(\mathrm{Res}^G_{w\mathfrak{S}} E(\phi, \pi')\right)_{\pi' = w\pi} =: {*e_{\mathfrak{C}}}(\phi, \pi),$$

where $*$ is a suitable scalar.

The study of this integral is delicate in the number field case because it does not seem to converge absolutely. Suppose thus that k is a number field. We consider the operator $\Delta(f_0)$ (see Chapter III) associated with the function f_0, element of $H^R_{\mathfrak{X}}$, defined by $f_0(\pi) = (\lambda_\pi, \lambda_\pi)$ where λ_π is the unique element of $\mathrm{Rat}\, M \otimes_{\mathbf{Z}} \mathbb{C}$ such that $\lambda_\pi \otimes \pi_0 \simeq \pi$ (π_0 is the canonical element of $\mathfrak{P}$) and where (,) is the $\mathbb{C}$-linear continuation of an invariant scalar product on $\mathrm{Rat}\, M \otimes_{\mathbf{Z}} \mathbb{R}$. The spectrum of this operator is real, and for a point of continuity T of this spectrum we write q_T for the projection onto the spectral part greater than or equal to $-T$. In fact we begin by describing:

$$L^2(G(k)\backslash\mathbf{G})_{\mathfrak{C}} \cap q_T L^2(G(k)\backslash\mathbf{G})_\xi =: L^2(G(k)\backslash\mathbf{G})_{\mathfrak{C},T}$$

which is neither more nor less than $q_T L^2(G(k)\backslash\mathbf{G})_{\mathfrak{C}}$. For this, we give an *ad hoc* definition ([L] first notation in 7.6 and V.2.3) of $L^2(G(k)\backslash\mathbf{G})_{\mathfrak{C},T}$:

we write $\mathrm{proj}_{\mathfrak{C},T}$ for the orthogonal projection onto this space. We then easily check that removing a subset of measure zero from

$$(**) \qquad \{\pi \in \mathfrak{G} \mid \mathrm{Re}\,\pi = o(\mathfrak{G}) \ \text{ and } \ \|\,\mathrm{Im}\,\lambda_\pi\|^2 < T + \|o(\mathfrak{G})\|^2\},$$

the map $\pi \mapsto e_{\mathfrak{C}}(\phi, \pi)$, for a fixed, R-Paley-Wiener ϕ, is holomorphic: we write $A_{\mathfrak{C},\pi}$ for the vector space generated as ϕ varies. Still removing a set of measure zero, we show that $A_{\mathfrak{C},\pi}$ is equipped with a Hilbertian scalar product characterised by the formula:

$$\langle e_{\mathfrak{C}}(\phi,\pi), e_{\mathfrak{C}}(\phi',\pi)\rangle = \langle \theta_\phi, e_{\mathfrak{C}}(\phi',\pi)\rangle,$$

for every Paley-Wiener ϕ and R-Paley-Wiener ϕ' (see V.3.9 which is 'greatly inspired' by [L], 7.5).

We thus show that $\pi \mapsto A_{\mathfrak{C},\pi}$ is a preHilbertian field which we easily transform, by completion, into a Hilbertian field. We then have a 'spectral' description of $L^2(G(k)\backslash\mathbf{G})_{\mathfrak{C},T}$ (see V.3.11), and it is with the help of this description and simple arguments from linear algebra that we show that $\pi \mapsto e_{\mathfrak{C}}(\phi,\pi)$ is holomorphic at every point of the set (**) above (see [L], proof of 7.6 and V.3.11). It is then no longer very difficult (see V.3.11) to prove the formula:

$$\mathrm{proj}_{\mathfrak{C},T}\,\theta_\phi = \int_{\substack{\pi\in\mathfrak{G},\,\mathrm{Re}\,\pi=o(\mathfrak{G})\\ \|\,\mathrm{Im}\,\lambda_\pi\|^2<T+\|o(\mathfrak{G})\|^2}} e_{\mathfrak{C}}(\phi,\pi)\,d\pi.$$

We then prove that:

$$q_T\theta_\phi = \sum_{\mathfrak{C}\in[S_{\mathfrak{X}}]} \mathrm{proj}_{\mathfrak{C},T}\,\theta_\phi,$$

almost all the terms which occur are zero since we are supposing that ϕ is $\mathbf{K}$-finite in the definition of R-Paley-Wiener. For this, it is necessary to compare, and thus in particular to compute, the scalar product of $q_T\theta_\phi$ and $\mathrm{proj}_{\mathfrak{C},T}\,\theta_\phi$ against all the pseudo-Eisenstein series (see V.2.9 and V.3.4): this uses Corollary 1 of [L], 7.4, which becomes Lemmas V.2.10 and V.3.5 below. We thus decompose not $L^2(G(k)\backslash\mathbf{G})_{\mathfrak{X}}$ but $q_T L^2(G(k)\backslash\mathbf{G})_{\mathfrak{X}}$ (see [L] 7.6 and V.3.12). Finally, letting T tend to infinity, we check that the slowly decreasing functions $\mathrm{proj}_{\mathfrak{C},T}\,\theta_\phi$ (for ϕ fixed and R-Paley-Wiener) on $G(k)\backslash\mathbf{G}$, possess a limit in $L^2(G(k)\backslash\mathbf{G})_{\mathfrak{X}}$ ([L] 7.6 and 7.7 and V.3.14); the closed space generated by these limits is $L^2(G(k)\backslash\mathbf{G})_{\mathfrak{C}}$ and we obtain the desired decomposition. It is more or less immediate that $L^2(G(k)\backslash\mathbf{G})_{\mathfrak{C}}$ meets the discrete spectrum if and only if the elements of $\mathfrak{C}$ are reduced to a point (i.e. $\mathrm{proj}_{\mathfrak{C},T}$ is defined without integration). This space is then entirely contained in the discrete spectrum and we say that $\mathfrak{C}$ is the singular class attached to this space. In general, as a representation of $\mathbf{G}$, this space is not irreducible, it is even of infinite length (see the example of G_2).

The goal of Chapter VI is to describe $L^2(G(k)\backslash\mathbf{G})_{\mathfrak{C}}$ when this space does not belong to the discrete spectrum, with the help of a part of the discrete spectrum of a standard Levi of G. To avoid non-conceptual difficulties, we continue to suppose here that k is a number field. The result is then as follows.

We show that there exists a pair (L, δ), unique up to association, where $\mathbf{L}$ is a standard Levi of G and δ is a subspace of the spectrum of $\mathbf{L}$ of the form $L^2(L(k)\backslash\mathbf{L})_{\mathfrak{C}_L}$ (where $\mathfrak{C}_L$ is a singular class, as above) characterised by:

> $\forall \mathfrak{G}_L \in \mathfrak{C}_L$, $\mathfrak{G}_L$ is reduced to a point (see above), denoted by π_L, and $\pi_L \otimes X_L^G$ is an element of $\mathfrak{C}$ (where X_L^G is the analogue of X_M^G defined at the beginning of the introduction) (see ([L], 7.2) and VI.1.9).

Then we show that $L^2(G(k)\backslash\mathbf{G})_{\mathfrak{C}}$ is generated by the integrals over the imaginary axis of X_L^G of Eisenstein series of $\mathbf{L}$ to $\mathbf{G}$ associated with automorphic forms for $\mathbf{L}$ contained in δ. The integral is just a limit in the L^2 sense explained above (see VI.2.2 and the formulation given by Arthur at Corvallis ([A1]) of Langlands' results). In fact, this description of $L^2(G(k)\backslash\mathbf{G})_{\mathfrak{C}}$ is, as Langlands explains in Appendix II of his description, a consequence for all general π of the spaces $A_{\mathfrak{C},\pi}$ introduced above (see [L], 7.4 and VI.2.4) and of the functional equation for the residues of Eisenstein series (see VI.1.5). We show in particular that $A_{\mathfrak{C},\pi}$ (for general π) can be identified with the space of Eisenstein series from $\mathbf{L}$ to $\mathbf{G}$ constructed from automorphic forms contained in δ. For this, we use relatively precise information about the locations of the poles of the residues of Eisenstein series, $\mathrm{Res}_{\mathfrak{G}}^{G} E(\phi, \pi)$ defined above, which is [L], corollary to 7.6 and VI.1.2. Langlands uses this information in the proof of 7.7; we use it in the 'reduction' to the Levi (see VI.1.4) which is implicit in ([L], 7.4 and after).

In Appendix I, we show that the covering $\mathbf{G} \to G(A)$ is split over every subgroup $U(\mathbb{A})$, where U is the unipotent radical of a parabolic subgroup of G. This is easy to prove if the characteristic of k does not divide the number of sheets of the covering. It is much less easy in the general case. The proof was communicated to us by P. Deligne.

In Appendix II we show that if k is a function field, every automorphic form on $G(k)\backslash\mathbf{G}$ is a linear combination of derivatives of Eisenstein series (see 1.4). The reasoning is as follows. Let ϕ be an automorphic form on $G(k)\backslash\mathbf{G}$. For every pseudo-Eisenstein series θ_Φ, the integral

$$I(\phi, \Phi) = \int_{G(k)\backslash\mathbf{G}} \phi(g)\theta_\Phi(g)\,\mathrm{d}g$$

is convergent. It is easy enough to express this by means of cuspidal components of constant terms of ϕ (see 2.2). Then the implication

$$\theta_\Phi = 0 \Rightarrow I(\phi, \Phi) = 0$$

imposes on these components the condition that they satisfy a system of linear equations. In particular, the cuspidal components of derivatives of Eisenstein series are solutions of this system. We show that every solution of this system comes from such a function. The argument is algebraic: it is here that we use the hypothesis that k is a function field. The final result is an easy consequence of this.

Appendix III studies the part of the discrete spectrum of a group of type G_2 coming from a minimal Levi and its trivial representation. Langlands showed that the subspace of vectors of this spectrum invariant under the maximal compact subgroup is of dimension 2. We remove this hypothesis of invariance (we do keep it for the archimedean places; we suppose k is a number field). The result is bizarre...

In Appendix IV, we study the modifications necessary to the theory when G is no longer supposed (algebraically) connected. We are thinking, for example, of the case of the orthogonal group. Essentially, we can say that the results are exactly the same as in the connected case, as long as the split ranks of the center of G and of its connected component of the identity G^0 are equal. In order to perform the inductions, we are led to define the notion of Levi subgroup in such a way that these subgroups satisfy these conditions. For this, if M^0 is a Levi subgroup of G^0, we associate with it a Levi subgroup M of G which is by definition the commutator in G of the largest split central torus in M^0. With these definitions, the results of Chapters 5 and 6 can be extended to the non-connected case. There are however some problems which must be resolved. For example, over a local field, there does not always exist a compact maximal subgroup intersecting every connected component. There is no Iwasawa decomposition in this case. But we show that this does not seriously perturb the proofs. If G does not satisfy the condition mentioned above, we must introduce the group G', the commutator in G of the largest split central torus in G^0. This group satisfies the desired condition and the spectral decomposition for the group G can be deduced by induction from the one for G'. Note, for example, that for such a group, the 'discrete spectrum' is reduced to $\{0\}$!

Notation

k, $\mathbb{A}$, $\mathbb{A}_\infty$, $\mathbb{A}_f$, q, G, K, $\mathbf{G}$, i_G I.1.1
P_0, M_0, Z_G, $\mathbf{K}$, $\mathbf{M}$, $Z_{\mathbf{G}}$, $\mathrm{Rat}(M)$, M^1, $\mathbf{M}^1$, $|\chi|$, $\mathfrak{a}_M^*$, $\mathrm{Re}\,\mathfrak{a}_M^*$, $\mathfrak{a}_M$, $\mathrm{Re}\,\mathfrak{a}_M$, X_M, $X_M^{\mathbf{G}}$, $\mathrm{Re}\,X_M$, $\mathrm{Im}\,X_M$, $\kappa : \mathfrak{a}_M^* \to X_M$, $\log_M$,
$m_P : \mathbf{G} \to \mathbf{M}^1 \backslash \mathbf{M}$ I.1.4
$Z_{\mathbf{G}}^1$ I.1.5
T_0, $R(T_0, G)$, $R^+(T_0, G)$, Δ_0, $R(T_0, M)$, Δ_0^M, T_M, $R(T_M, G)$, Δ_M, $(\mathfrak{a}_M^{M'})^*$, I.1.6
W, W_M, $W(M)$ I.1.7
$\check{\alpha}$, α^* I.1.11
ρ_0, ρ_P I.1.13
$A_{\mathbf{M}}$, $\mathbf{M}^c$, S, S^P I.2.1
$\|g\|$ I.2.2
δ, $\mathcal{U}$ I.2.3
$L^2(G(k)\backslash\mathbf{G})_\xi$ I.2.5
ϕ_P I.2.6
$\mathbf{M}_0(P, t)$ I.2.7
$s\phi$ I.2.9
$\hat{\tau}_P$, $\Lambda^T\phi$ I.2.13
$\mathfrak{z}$, $\mathfrak{z}^{\mathbf{M}}$, ϕ_k, $A(U(\mathbb{A})M(k)\backslash\mathbf{G})$, $A(U(\mathbb{A})M(k)\backslash\mathbf{G})_\xi$ I.2.17
ϕ^{cusp}, $A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_\xi$ I.2.18
$X_M(A_{\mathbf{M}})$, $\mathfrak{z}(A_{\mathbf{M}}; \Lambda, N)$, $\mathbb{C}[\mathrm{Re}\,\mathfrak{a}_M]$, $\mathfrak{q}_M$ I.3.1
$A(U(\mathbb{A})M(k)\backslash\mathbf{G})_Z$ I.3.2
$\Pi_0(\mathbf{M})_\xi$, $\Pi_0(\mathbf{M})$, $A_0(U(\mathbb{A})M(k)\backslash\mathbf{G})_\pi$, χ_π, $\mathrm{Re}\,\pi$, $\mathrm{Im}\,\pi$, $-\pi$, $-\overline{\pi}$ I.3.3
$D(M, \phi)$, $\Pi_0(\mathbf{M}, \phi)$ I.3.3 and I.3.5

$C_0(U(\mathbb{A})M(k)\backslash \mathbf{G})$ I.3.4
ϕ_P^{cusp} I.3.5
$A(d, R, Y)$ I.4.2
$s(\varphi, r)$ I.4.3
$A((V_P, \Gamma_P, N_P)_{P_0 \subset P \subset G})$ I.4.4
$\mathfrak{w}_{\check{\alpha}}$ I.4.11
$A(U(\mathbb{A})M(k)\backslash \mathbf{G})_\pi$ II.1.1
$\text{Fix}_{X_M^{\mathbf{G}}} \mathfrak{P}$ II.1.1
$P_{(M,\mathfrak{P})}$ II.1.2
$\epsilon F(\phi)$ II.1.3
$P_{(M,\mathfrak{P})}^R$ II.1.4
$E(\phi, \pi)$ II.1.5
$M(w, \pi)$ II.1.6
$W(M, M')$ II.1.7
θ_ϕ II.1.10
$(M, \mathfrak{P}) \sim (M', \mathfrak{P}')$ II.2.1
$\mathfrak{P}_{M'}$, $P_{(M,\mathfrak{P}_{M'})}^R$ II.2.2
$\Theta_{\mathfrak{X}}^R$, $L_{\mathfrak{X}}^2$, $H_{\mathfrak{X}}^R$, $\tilde{\mathfrak{X}}^R$, $H_{\mathfrak{X},\mathbb{C}}^R$, $f_{M,\pi}$ III.1.1
$\Delta_\theta(f)$ III.1.2
$H_{\mathfrak{X},b}^R$ III.1.4
$\Delta(f)$ III.1.4 and III.2.1
λ_π, Δ III.1.5
p_T, q_T III.1.6 and V.2.8
Θ_ξ^R, H_ξ^R, Θ_ξ, $H_{\xi,b}^R$, $H_{\xi,\mathbb{C}}^R$ III.1.7
$\Theta_{\eta^\bullet}^{M,G}$ III.2.3
$H_\xi^{\exp}$ III.2.4
$A^2(G(k)\backslash \mathbf{G})_\xi$ III.3.1
$N(G)$ III.3.2
$\mathfrak{F}$, A_ξ, L_ξ^2, $L_{\xi,\text{loc}}^2$, $\mathscr{H}^{\mathfrak{F}}$ IV.1.1
$\mathscr{C}_{\mathfrak{P}}$ IV.1.7
$\mathscr{C}$ IV.2.1
k_h, $\Lambda_1^T k_h$ IV.2.5
E^T IV.3.2
F^T IV.3.3
V_h, e^T, $V_{w,h}$ IV.3.5
$\tilde{E}$ IV.3.6
$\mathfrak{X}$, ξ V.1.1
$\text{Stab}(M, \mathfrak{P})$ V.1.1
$\mathfrak{G}^0$, $\tilde{\mathfrak{G}}^0$ V.1.1
$H_{\alpha^\bullet}$ V.1.1
$S_{(M,\mathfrak{P})}$, $S_{\mathfrak{X}}$ V.1.1

$S_{\mathfrak{X}}^{\mathfrak{F}}$ V.1.1
$\mathfrak{G}_{\leq T} = T$ V.1.5(a)
λ_π V.1.5(a)
$P_{\mathfrak{X}}^{\mathfrak{F}}$ V.2.1
R, $P_{(M,\mathfrak{P})}^{R,\mathfrak{F}}$ V.2.1
$o(\mathfrak{G})$ V.2.1
$T - \mathfrak{F}$-general point V.2.1
$A(\phi', \phi)$ V.2.1 and V.3.1
Norm $\mathfrak{G}$, $\mathrm{Res}_{\mathfrak{G}}^{G}$ V.2.2
q_T V.2.8
H^R V.2.9
$\mathfrak{G}_{\mathfrak{C}}$, $z(\mathfrak{G})$ V.3.1
$\mathfrak{G} \sim \mathfrak{G}'$ V.3.1
$[S_{\mathfrak{X}}]$, $[S_{\mathfrak{X}}^{\mathfrak{F}}]$ V.3.1
$r_{\mathfrak{C}}(\phi', \phi)$, $m_{\mathfrak{C},T}(\phi', \phi)$ V.3.1
$\mathrm{proj}_{\mathfrak{C},T}^{\mathfrak{F}}$ V.3.2 and V.3.3
$e_{\mathfrak{C}}(\phi, \pi)$, $A_{\mathfrak{C},\pi}^{\mathfrak{F}}$ V.3.2
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