I

Hypotheses, Automorphic Forms, Constant Terms

I.1. Hypotheses and general notation

I.1.1. Definitions

Let k be a global field and $\mathbb{A}$ be the ring of adeles of k. For a finite place v of k, we write ϑ_v for the ring of integers. Let $\mathbb{A}_f$ be the ring of finite adeles of k and $\mathbb{A}_\infty = \prod k_v$, the product being over the archimedean places. If k is a function field, let q be the number of elements of its field of constants.

Let G be a connected reductive algebraic group defined over k. Fix an embedding into a linear group as follows. First choose an embedding $i'_G : G \hookrightarrow GL_n$, defined over k, with closed image. Then $i_G : G \hookrightarrow GL_{2n}$ is defined by

$$i_G(g) = \begin{pmatrix} i'_G(g) & 0 \\ 0 & {}^t i'_G(g)^{-1} \end{pmatrix}.$$

There exists a finite set S of places of k, containing the archimedean places, such that the image of i_G is defined and smooth over $\vartheta^S := \prod_{v \notin S} \vartheta_v$ (see [Sp] §4.9). For $v \notin S$, this allows us to define the group $G(\vartheta_v)$ of points with values in ϑ_v. For almost all $v \notin S$, this is a maximal compact subgroup of $G(k_v)$ (see [Sp] p.18, 1.3 and what follows). We fix a compact maximal subgroup K of $G(\mathbb{A})$ such that $K = \prod_v K_v$, product over all places of k, where K_v is a maximal compact subgroup of $G(k_v)$. We suppose, as we may, that $K_v = G(\vartheta_v)$ for almost all finite places. We will impose further properties on K in I.1.4.

Let $\mathbf{G}$ be a topological group which is a finite central covering of $G(\mathbb{A})$ (see [M3]), i.e. there exists a surjective morphism pr of topological groups, whose kernel $\mathbf{N}$ is a finite subgroup of the centre of $\mathbf{G}$, pr being a topological covering. This last condition can be interpreted as follows: there exists an open neighborhood U of the unit element of $G(\mathbb{A})$ and an isomorphism j of $\mathrm{pr}^{-1}(U)$ onto $U \times \mathbf{N}$ which makes the following diagram commute:

$$\begin{array}{ccc} \mathrm{pr}^{-1}(U) & \xrightarrow{j} & U \times \mathbf{N} \\ & \mathrm{pr} \searrow \quad \swarrow \mathrm{pr}_1 & \\ & U & \end{array}$$

where pr_1 denotes projection onto the first factor.

I.1.2. Description of G

We begin with a remark.

Remark *There exists a finite set S of places of k such that for all $v \notin S$, K_v lifts into* $\mathbf{G}$.

Proof Fix U as above. We require that j satisfy the condition $j(1_{\mathbf{G}}) = 1_{\mathbf{N}} \times 1_{G(\mathbb{A})}$, where 1 with an index indicates the unit element of the group in the index. Set $\tilde{U} = j^{-1}(U \times 1_{\mathbf{N}})$. As j is a homeomorphism, $\tilde{U}$ is an open set of $\mathrm{pr}^{-1}(U)$ containing $1_{\mathbf{G}}$; thus it is a neighborhood of $1_{\mathbf{G}}$ on which the restriction of pr coincides with j and induces an isomorphism onto U. Fix an open neighborhood $\tilde{U}'$ of $1_{\mathbf{G}}$ in $\mathbf{G}$, contained in $\tilde{U}$, such that $\tilde{U}'\tilde{U}' \subset \tilde{U}$. Set $U' = \mathrm{pr}(\tilde{U}')$. Let S be a finite set of places of k and U'_S a neighborhood of the unit element of $\prod_{v \in S} G(k_v)$ such that

$$U' \supset U'_S \times \prod_{v \notin S} K_v;$$

the existence of S and U'_S results from the description of the adelic topology of $G(\mathbb{A})$. For all $v \notin S$, K_v can be identified with a subgroup of U' and we set $\tilde{K}_v = \mathrm{pr}^{-1}(K_v) \cap \tilde{U}$; we note that $\tilde{K}_v$ is contained in $\tilde{U}'$. By the preceding discussion, pr induces an isomorphism of $\tilde{K}_v$ onto K_v. In particular, if v is finite, $\tilde{K}_v$ is open and compact. The property of $\tilde{U}'$ then implies that $\tilde{K}_v$ is a subgroup of $\mathbf{G}$, which gives the desired result. □

The lifting defined above is clearly not canonical and has no reason to be unique. It does however satisfy the following property:

(1) for every open set $\mathbf{U}$ of $\mathbf{G}$ containing $1_{\mathbf{G}}$, there exists a finite set of places S such that $\mathbf{U}$ contains $\prod_{v \notin S} \tilde{K}_v$.

Two systems of liftings (i.e. liftings of K_v for almost all v) satisfying this hypothesis coincide almost everywhere.

For a place v of k, we denote by $\mathbf{G}_v$ the inverse image of $G(k_v)$ under pr. It is clear that $\mathbf{G}_v$ equipped with the topology induced by that of $\mathbf{G}$ is a topological group. There is an exact sequence realising $\mathbf{G}_v$ as a covering of $G(k_v)$:

$$1 \longrightarrow \mathbf{N} \longrightarrow \mathbf{G}_v \xrightarrow{\text{pr}} G(k_v) \longrightarrow 1$$

For almost all v, fix a lifting $\tilde{K}_v$ of K_v such that we obtain a system of liftings satisfying (1) above, and set

$$\prod_v{}' \mathbf{G}_v = \{(g_v)_{v \text{ a place of } k}; g_v \in \mathbf{G}_v \text{ and for almost all } v, g_v \in \tilde{K}_v\}.$$

This is independent of the choice of liftings. Put the adelic topology on $\prod_v' \mathbf{G}_v$. This group contains in its centre the group

$$\bigoplus_v \mathbf{N} = \{(n_v)_{v \text{ a place of } k}; n_v \in \mathbf{N} \text{ and } n_v = 1 \text{ for almost all } v\}.$$

Define

$$(\bigoplus_v \mathbf{N})^1 = \{(n_v) \in \bigoplus_v \mathbf{N};\ \prod n_v = 1\}.$$

We then have the isomorphism

$$\prod_v{}' \mathbf{G}_v / (\bigoplus_v \mathbf{N})^1 \xrightarrow{\sim} \mathbf{G}. \tag{2}$$

Conversely, let $\mathbf{N}$ be a finite abelian group and for every place v of k, fix a topological group $\mathbf{G}_v$ which is a central extension of $G(k_v)$ by $\mathbf{N}$, i.e. such that

$$1 \longrightarrow \mathbf{N} \longrightarrow \mathbf{G}_v \xrightarrow{\text{pr}_v} G(k_v) \longrightarrow 1.$$

We suppose that for almost all finite places of k, there exists a compact open subgroup $\tilde{K}_v$ of $\mathbf{G}_v$ such that the restriction of pr_v to $\tilde{K}_v$ is an isomorphism from $\tilde{K}_v$ onto K_v. We define the topological group $\prod_v' \mathbf{G}_v$ as above, as well as its subgroup $(\bigoplus_v \mathbf{N})^1$. Then

$$\mathbf{G} := (\prod_v{}' \mathbf{G}_v) / (\bigoplus_v \mathbf{N})^1$$

is a natural extension of $G(\mathbb{A})$ by $\mathbf{N}$.

I.1.3. The covering $\mathbf{G}_v \xrightarrow{\text{pr}} G(\mathbb{A})$: hypotheses and properties

Suppose that

(1) $G(k)$ lifts to a subgroup of $\mathbf{G}$.

Fix once and for all a lifting of it which we also denote by $G(k)$. Note that:

(2) if C is a commutative subgroup of $G(\mathbb{A})$ and $\mathbf{C}$ is its inverse image, then the centre of $\mathbf{C}$ contains the inverse image of

$$C^{|\mathbf{N}|}(:= \{c^{|\mathbf{N}|}; c \in C\}),$$

where $|\mathbf{N}|$ is the number of elements of $\mathbf{N}$.

The proof of this is elementary.

We need to lift the unipotent subgroups of $G(\mathbb{A})$. More precisely, let P be a parabolic subgroup of G (defined over k) and U its unipotent radical. In the appendix we prove that

(3) $U(\mathbb{A})$ lifts canonically into $\mathbf{G}$.

We still use $U(\mathbb{A})$ to denote the image of this lifting. Then $U(\mathbb{A})$ is the adelic product of subgroups defined for every place v of k and isomorphic to $U(k_v)$. Moreover (3) implies that

(4) The inverse image of the normaliser of $U(\mathbb{A})$ in $G(\mathbb{A})$ is the normaliser of $U(\mathbb{A})$ in $\mathbf{G}$.

We will return to the properties of the lifting $\mathrm{pr} : \mathbf{G} \to G(\mathbb{A})$ later on (see I.1.5).

I.1.4. Levi subgroups and characters

Let $k, A, G, \mathbf{G}, \mathrm{pr}, K$ be as previously introduced (see I.1.1). We denote by $\mathbf{K}$ the inverse image of K in $\mathbf{G}$. Let P be a parabolic subgroup of G (i.e. G/P is a complete algebraic variety), defined over k, of unipotent radical U, and let M be a Levi subgroup of P. Let $\mathbf{M}$ denote the inverse image of $M(\mathbb{A})$ in $\mathbf{G}$. We still use $U(\mathbb{A})$ to denote the canonical lifting of $U(\mathbb{A})$ into $\mathbf{G}$ (see I.1.3 (3)). Fix a parabolic subgroup P_0 of G, defined over k and minimal, and a Levi subgroup M_0 of P_0, defined over k. We use the phrase 'standard parabolic subgroup of G' to denote any parabolic subgroup of G defined over k and containing P_0. By a 'standard Levi subgroup of G' we mean any Levi subgroup, containing M_0, of a standard parabolic subgroup of G. Every standard parabolic subgroup possesses a unique standard Levi subgroup.

It is possible to require that the group K have the following properties:

(i) $G(\mathbb{A}) = P_0(\mathbb{A})K$;

(ii) for every standard parabolic subgroup $P = MU$ of G, the equality $P(\mathbb{A}) \cap K = (M(\mathbb{A}) \cap K)(U(\mathbb{A}) \cap K)$ is satisfied, and $M(\mathbb{A}) \cap K$ is still a compact maximal subgroup of $M(\mathbb{A})$.

Thanks to (ii), the choice of K also fixes a choice of compact maximal subgroup of $M(\mathbb{A})$ for every standard Levi M.

Remark Hypothesis (ii) is useful but not indispensable. If it was not satisfied, we would have to choose a compact maximal subgroup of $M(\mathbb{A})$ containing the image in $M(\mathbb{A}) \simeq P(\mathbb{A})/U(\mathbb{A})$ of $P(\mathbb{A}) \cap K$.

Denote by Z_G the centre of G, by $Z_{\mathbf{G}}$ that of $\mathbf{G}$ and for M a Levi subgroup of G, denote by Z_M the centre of M and by $Z_{\mathbf{M}}$ that of $\mathbf{M}$.

Remark In general, $Z_{\mathbf{M}}$ is not equal to the inverse image of $Z_M(\mathbb{A})$ under pr. However, we will show as in I.1.3 (2), that $Z_{\mathbf{M}}$ contains $\mathrm{pr}^{-1}(Z_M(\mathbb{A})^{|N|})$.

Fix a standard Levi M of G; let $\mathrm{Rat}(M)$ denote the group of rational characters of M (i.e. the homomorphisms as algebraic groups of M into the multiplicative group $\mathbb{G}_m$). Set

$$\mathfrak{a}_M^* = \mathrm{Rat}(M) \bigotimes_{\mathbb{Z}} \mathbb{C}, \ \mathfrak{a}_M = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(M), \mathbb{C}).$$

The complex vector spaces $\mathfrak{a}_M$ and $\mathfrak{a}_M^*$ are duals of each other. Note that they are naturally equipped with $\mathbb{Q}$-structures: if we set

(1) $$\mathfrak{a}_{M,\mathbb{Q}}^* = \mathrm{Rat}(M) \bigotimes_{\mathbb{Z}} \mathbb{Q}, \ \mathfrak{a}_{M,\mathbb{Q}} = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(M), \mathbb{Q}),$$

then obviously

$$\mathfrak{a}_M^* = \mathfrak{a}_{M,\mathbb{Q}}^* \bigotimes_{\mathbb{Q}} \mathbb{C}, \ \mathfrak{a}_M = \mathfrak{a}_{M,\mathbb{Q}} \bigotimes_{\mathbb{Q}} \mathbb{C}.$$

We will also use the real spaces

$$\mathrm{Re}\,\mathfrak{a}_M^* = \mathrm{Rat}(M) \bigotimes_{\mathbb{Z}} \mathbb{R}, \ \mathrm{Re}\,\mathfrak{a}_M = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(M), \mathbb{R}).$$

Let $\chi \in \mathrm{Rat}(M)$: for every place v of k, χ defines an algebraic character denoted by χ_v of $M(k_v)$ into k_v^*. We define $|\chi|$, a continuous homomorphism of $M(\mathbb{A})$ into $\mathbb{C}^*$, by

$$\forall m = (m_v) \in M(\mathbb{A}), \ m^{|\chi|} = \prod_v |m_v^{\chi_v}|_v,$$

where $|\ |_v$ is the absolute value of k_v (in general, if H is a group, h an element of H and χ a character of H, we write h^χ for the value of χ at the point h, which induces us to write the group law on the set of characters of H additively). Let

$$M^1 = \bigcap_{\chi \in \mathrm{Rat}(M)} \mathrm{Ker}|\chi|;$$

it is a normal subgroup of $M(\mathbb{A})$, and let X_M be the group of continuous homomorphisms of $M(\mathbb{A})$ into $\mathbb{C}^*$ which are trivial on M^1.

Given the importance of the group X_M in what follows, we give a further description of it here.

(2) Let $\lambda \in X_M$. Then there exist $\chi_1, \ldots, \chi_R 2 \in \mathrm{Rat}(M)$ and $s_1, \ldots, s_R \in \mathbb{C}$ such that

$$\forall m \in M(\mathbb{A}),\ m^{\lambda} = (m^{|\chi_1|})^{s_1} \ldots (m^{|\chi_R|})^{s_R}.$$

Proof The group $\mathrm{Rat}(M)$ is a free $\mathbb{Z}$-module since M is connected. Fix a basis $\chi_1, \ldots, \chi_R$ of this $\mathbb{Z}$-module and denote by V the subgroup of $\mathbb{R}_+^*$ which is the image of $\mathbb{A}^*$ under the absolute value map (if k is a number field then $V = \mathbb{R}_+^*$ and if k is a function field then $V = q^{\mathbb{Z}}$). Define a map

$$j : M(\mathbb{A}) \to V^R$$

by $j(m) = (m^{|\chi_1|}, \ldots, m^{|\chi_R|})$. Its kernel is M^1 since

$$M^1 = \bigcap_{i=1}^{R} \mathrm{Ker}\,|\chi_i|.$$

Its image is all of V^R if k is a number field, a subgroup of V^R of finite index if k is a function field. For suppose first that M is a split torus. Then the map

$$M(\mathbb{A}) \longrightarrow \mathbb{A}^{*R}$$

$$m \longmapsto (m^{\chi_1}, \ldots, m^{\chi_R})$$

is surjective and the assertion is clear. In the general case, it suffices to prove the same assertion for the subgroup $j(T_M(\mathbb{A}))$, where T_M is the maximal split torus of Z_M. The set of restrictions $\{\chi_{i|T_M}; i = 1, \ldots, R\}$ does not generate $\mathrm{Rat}(T_M)$ but remains linearly independent and generates a subgroup of finite index of $\mathrm{Rat}(T_M)$, which concludes the argument.

We check that j defines a topological group isomorphism of $M(\mathbb{A})/M^1$ onto the image $j(M(\mathbb{A}))$. A continuous homomorphism of $M(\mathbb{A})/M^1$ into $\mathbb{C}^*$ thus comes from a continuous homomorphism of V^R into $\mathbb{C}^*$ and the latter is always of the form

$$(x_1, \ldots, x_R) \longmapsto x_1^{s_1} \ldots x_R^{s_R}$$

for some suitable $s_1, \ldots, s_R \in \mathbb{C}$. □

In other words, X_M can be realised as a quotient of $\mathfrak{a}_M^*$, i.e. there exists a surjective morphism of groups

(3) $$\kappa : \mathfrak{a}_M^* \to X_M.$$

The above argument proves that κ is bijective if k is a number field; if k is a function field, the kernel of κ is of the form $(2\pi i/\log q)L$, where L is a lattice of $\mathrm{Rat}(M) \otimes_{\mathbb{Z}} \mathbb{Q}$ containing the image of $\mathrm{Rat}(M)$ in this vector space. We obtain the same result in an equivalent way by the following considerations. There exists a natural map of $M(A)$ into $\mathfrak{a}_M$, denoted by $\log_M$, defined by

(4) $$\forall m \in M(\mathbb{A}),\ \forall \chi \in \mathrm{Rat}(M),\ \langle \chi, \log_M(m)\rangle = \log(m^{|\chi|}).$$

The kernel of $\log_M$ is precisely M^1 and $M(\mathbb{A})/M^1$ can be identified via $\log_M$ with a subgroup of $\mathfrak{a}_M$ and even of $\mathrm{Re}\,\mathfrak{a}_M$. The kernel of κ is the set of $\lambda \in \mathfrak{a}_M^*$ such that $e^{\langle \lambda, H\rangle} = 1$ for all H in the image of $\log_M$. If k is a number field, then $\log_M(M(\mathbb{A})/M^1) \simeq \mathrm{Re}\,\mathfrak{a}_M$, and κ is bijective. If k is a function field, $\log_M(M(\mathbb{A})/M^1)$ is a lattice of $\mathrm{Re}\,\mathfrak{a}_M$ contained in $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(M), (\log q)\mathbb{Z})$.

Let us return to the situation where k is arbitrary. Set

$$\mathrm{Re}\,X_M = \kappa(\mathrm{Re}\,\mathfrak{a}_M^*),\ \mathrm{Im}\,X_M = \kappa(\mathrm{i}\,\mathrm{Re}\,\mathfrak{a}_M^*).$$

It is a consequence of the above arguments that κ induces an isomorphism

(5) $$\mathrm{Re}\,\mathfrak{a}_M^* \simeq \mathrm{Re}\,X_M.$$

We check that $\mathrm{Re}\,X_M$ is the group of characters of $M(\mathbb{A})/M^1$ (i.e. of continuous homomorphisms of $M(\mathbb{A})/M^1$ into $\mathbb{C}^*$) with values in $\mathbb{R}_+^*$. The group X_M can be identified with the group of characters of $\mathbf{M}$ trivial on $\mathbf{M}^1$, where $\mathbf{M}^1$ is the inverse image of M^1. The group $\mathrm{Re}\,X_M$ can then be identified with the group of characters of $\mathbf{M}/\mathbf{M}^1$ with values in $\mathbb{R}_+^*$. As we will see later on, if we equip $\mathbf{M}^1$ with a Haar measure, the quotient $M(k)\backslash\mathbf{M}^1$ is of finite volume. Thus a character of $\mathbf{M}^1$ with values in $\mathbb{R}_+^*$ which is trivial on $M(k)$ is necessarily trivial. Then $\mathrm{Re}\,X_M$ can also be identified with the group of characters of $\mathbf{M}$ with values in $\mathbb{R}_+^*$ which are trivial on $M(k)$. By Lemma I.1.6 below, it can similarly be identified with the group of characters of $Z_{\mathbf{M}}$ with values in $\mathbb{R}_+^*$ which are trivial on $Z_M(k) \cap Z_{\mathbf{M}}$.

(6) We write $X_M^{\mathbf{G}}$ for the subgroup of X_M consisting of the characters of $\mathbf{M}/\mathbf{M}^1$ trivial on $Z_{\mathbf{G}}$.

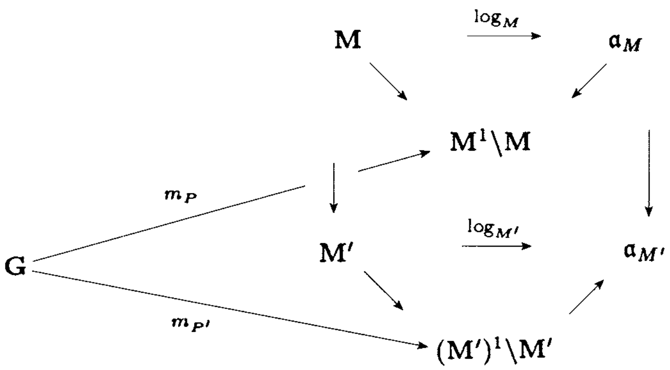
This group will play an essential role in our constructions. We will return to it later on (see I.1.6 (6)). Let us still write $\log_M$ for $\log_M$ composed with pr. Let p be the standard parabolic subgroup of Levi subgroup M and let U be its unipotent radical. Via the equality $\mathbf{G} = \mathbf{PK}$ (which is a consequence of I.1.4 (i)), we define a map

$$m_P : \mathbf{G} \longrightarrow \mathbf{M}^1\backslash\mathbf{M}$$

by $m_P(g) = \mathbf{M}^1 m$ if $g = umk$, where $u \in U(\mathbb{A})$, $m \in \mathbf{M}$, $k \in \mathbf{K}$. This is well-defined, for $\mathbf{M} \cap \mathbf{K}$ is contained in $\mathbf{M}^1$ and $\mathbf{M}^1$ is a normal subgroup of $\mathbf{M}$.

Let $M \subset M'$ be the standard Levis of parabolic subgroups $P \subset P'$;

the restriction map identifies $X_{M'}$ with a subgroup of X_M, so $\mathrm{Re}\, X_{M'}$ is naturally a vector subspace of $\mathrm{Re}\, X_M$ and there are maps $\mathfrak{a}^*_{M'} \hookrightarrow \mathfrak{a}^*_M$ and $\mathfrak{a}_M \to \mathfrak{a}_{M'}$. We also have a commutative diagram



I.1.5. Description of G continued

Set $Z^1_G = Z_G(\mathbb{A}) \cap G^1$, $Z^1_{\mathbf{G}} = Z_{\mathbf{G}} \cap \mathbf{G}^1 = Z_{\mathbf{G}} \cap pr^{-1}(Z^1_G)$.

Lemma *The subgroups $Z_G(k)Z_{\mathbf{G}}$ and $G(k)Z_{\mathbf{G}}$ of $\mathbf{G}$ are closed. The quotient $Z_G(k) \cap Z_{\mathbf{G}} \backslash Z^1_{\mathbf{G}}$ is compact.*

Proof We begin by showing that, for every place v of k, there exists a normal open subgroup of finite index $\mathbf{G}^\circ_v$ of $\mathbf{G}_v$ which commutes with $\mathrm{pr}^{-1}(Z_G(k_v))$. For $g \in \mathbf{G}_v$ and $z \in \mathrm{pr}^{-1}(Z_G(k_v))$, we have an equality

$$z^{-1} g z = \chi(g, z) g,$$

where χ is a bicharacter of $\mathbf{G}_v \times \mathrm{pr}^{-1}(Z_G(k_v)) \to \mathbf{N}$. As pr is a covering, there exist open neighborhoods U, V of 1 in $\mathbf{G}_v$, $\mathrm{pr}^{-1}(Z_G(k_v))$ respectively, such that χ is trivial on $U \times V$. Let $\mathbf{G}^N_v$, $\mathrm{pr}^{-1}(Z_G(k_v))^N$ be the set of Nth powers in $\mathbf{G}_v$, $\mathrm{pr}^{-1}(Z_G(k_v))$ respectively, where $N = |\mathbf{N}|$. The bicharacter χ is also trivial on $\mathbf{G}^N_v \times pr^{-1}(Z_G(k_v))$ and $\mathbf{G}_v \times \mathrm{pr}^{-1}(Z_G(k_v))^N$. Let U_0, V_0 be the normal subgroups generated by U and $\mathbf{G}^N_v$, V and $\mathrm{pr}^{-1}(Z_G(k_v))^N$ respectively. Then χ is trivial on $U_0 \times V_0$. We show easily that U_0, V_0, is of finite index in $\mathbf{G}_v$, $\mathrm{pr}^{-1}(Z_G(k_v))$ respectively. Let $z_1, \ldots, z_n$ be a set of representatives of $\mathrm{pr}^{-1}(Z_G(k_v))/V_0$. For $i = 1, \ldots, n$, let U_i be the kernel of the character $g \longmapsto \chi(g, z_i)$. Then U_i is open and of finite index in $\mathbf{G}_v$. Set $\mathbf{G}^\circ_v = \bigcap_{i=0}^n U_i$. This group is thus the desired normal open subgroup of $\mathbf{G}_v$ of finite index.

Choose a finite set S of places of k, containing the archimedean places, and large enough so that

(i) $\left(\prod_{v\in S} G(k_v)\prod_{v\notin S} K_v\right) G(k) = G(\mathbb{A})$;

(ii) for $v \notin S$, K_v lifts to a subgroup denoted by $\tilde{K}_v$ of $\mathbf{G}_v$.

Let $H = \prod_{v\in S} G(k_v)\prod_{v\notin S} K_v$ and let $\mathbf{H}$ be its inverse image in $\mathbf{G}$. We will show that

(*) $$\left(Z_G(k)\cap Z_{\mathbf{G}}\right)(\mathbf{H}\cap Z^1_{\mathbf{G}})$$

is of finite index in $Z^1_{\mathbf{G}}$.

Let Z^0_G be the connected component of the identity of Z_G. It is a torus. Set $Z^{01}_G = Z^0_G(\mathbb{A})\cap G^1$. This is actually the group which is deduced from $Z^0_G(\mathbb{A})$ as G^1 is deduced from $G(\mathbb{A})$. By the finiteness of the class number of $k, Z^0_G(k)(H\cap Z^{01}_G)$ is of finite index in Z^{01}_G. By Lemma 1 of Appendix IV, $Z^{01}_G(H\cap Z^1_G)$ is of finite index in Z^1_G. Thus $Z_G(k)(H\cap Z^1_G)$ is of finite index in Z^1_G. It thus suffices to show that $(Z_G(k)\cap Z_{\mathbf{G}})(\mathbf{H}\cap Z^1_{\mathbf{G}})$ is of finite index in $[Z_G(k)(\mathbf{H}\cap \mathrm{pr}^{-1}(Z^1_G))]\cap Z^1_{\mathbf{G}}$. Indeed, it suffices to show that $\mathbf{H}\cap Z^1_{\mathbf{G}}$ is of finite index in $\mathbf{H}\cap Z_G(k)Z^1_{\mathbf{G}}$. Let us consider an element h of this set. For $v\in S$, choose a group $\mathbf{G}^\circ_v$ as above. As $h\in \mathrm{pr}^{-1}(Z_G(\mathbb{A}))$, h commutes with $\mathbf{G}^\circ_v$. Let $v\notin S$. As $\tilde{K}_v\to K_v$ is an isomorphism and $h\in \mathbf{H}\cap\mathrm{pr}^{-1}(Z_G(\mathbb{A}))$, h commutes with $\tilde{K}_v$. As $Z_G(k)$ and $Z^1_{\mathbf{G}}$ commute with $G(k)$, h does as well. Thus h commutes with the normal subgroup generated by $G(k)$ and $\prod_{v\in S}\mathbf{G}^\circ_v\prod_{v\notin S}\tilde{K}_v$. Denote this subgroup by C. By (i), we see that C is of finite index in $\mathbf{G}$. For $g\in\mathbf{G}$, we write $h^{-1}gh = \chi(g,h)g$ as above. The map $h\mapsto \chi(.,h)$ is an injective map of

$$\mathbf{H}\cap Z^1_{\mathbf{G}}\backslash \mathbf{H}\cap Z_G(k)Z^1_{\mathbf{G}}$$

into the finite group of characters of $C\backslash\mathbf{G}$. Thus the above group is finite, which is what we wanted to prove.

Let $E = Z^0_G(k)\cap H$. By the Dirichlet unit theorem we see that

(iii) $E\backslash H\cap Z^1_G$ is compact;

(iv) E is the product of a finite group by a finitely generated free group.

Let E^N be the subgroup of N-th powers of E. It can be identified with a subgroup of $\mathbf{H}\cap Z^1_{\mathbf{G}}$. The natural map

$$E^N\backslash\mathbf{H}\cap Z^1_{\mathbf{G}} \longrightarrow E^N\backslash H\cap Z^1_G$$

is proper. By (iii) and (iv), the right-hand space is compact, so the left-hand one must be as well. Thus, we can find a compact subset Γ of $\mathbf{H}\cap Z^1_{\mathbf{G}}$ such that $\mathbf{H}\cap Z^1_{\mathbf{G}} = E^N\Gamma$. Then

$$(Z_G(k)\cap Z_{\mathbf{G}})(\mathbf{H}\cap Z^1_{\mathbf{G}}) = (Z_G(k)\cap Z_{\mathbf{G}})\Gamma$$

and by (*), we see that $Z_G(k)\cap Z_{\mathbf{G}}\backslash Z^1_{\mathbf{G}}$ is compact. The groups

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More information

$Z_G(k)Z^1_{\mathbf{G}}$, $G(k)Z^1_{\mathbf{G}}$ contain

$$Z_G(k)(Z_G(k)\cap Z_{\mathbf{G}})(\mathbf{H}\cap Z^1_{\mathbf{G}}) = Z_G(k)\Gamma,$$

$$G(k)(Z_G(k)\cap Z_{\mathbf{G}})(\mathbf{H}\cap Z^1_{\mathbf{G}}) = G(k)\Gamma,$$

respectively, as subgroups of finite index. The last groups are closed since they are products of a discrete set by a compact set. Thus, $Z_G(k)Z^1_{\mathbf{G}}$ and $G(k)Z^1_{\mathbf{G}}$ are closed. Let us use the construction of the proof of I.1.4 (2) for $M = G$. We have a continuous map

$$j\circ \mathrm{pr} : \mathbf{G} \longrightarrow V^R$$

whose kernel is $\mathbf{G}^1$. We thus have the following diagram of continuous maps

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{G}^1 & \longrightarrow & \mathbf{G} & \longrightarrow & j\circ\mathrm{pr}(\mathbf{G}) & \longrightarrow & 1\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & G(k)Z^1_{\mathbf{G}} & \longrightarrow & G(k)Z_{\mathbf{G}} & \longrightarrow & j\circ\mathrm{pr}(Z_{\mathbf{G}}) & \longrightarrow & 1
\end{array}$$

We check that $j\circ\mathrm{pr}(Z_{\mathbf{G}})$ is closed in $j\circ\mathrm{pr}(\mathbf{G})$ and that the map $Z_{\mathbf{G}} \to j\circ\mathrm{pr}(Z_{\mathbf{G}})$ admits a continuous section. Since $G(k)Z^1_{\mathbf{G}}$ is closed in $\mathbf{G}^1$, we see that $G(k)Z_{\mathbf{G}}$ is closed in $\mathbf{G}$. We argue similarly for $Z_G(k)Z_{\mathbf{G}}$. □

I.1.6. Roots and coroots

We have already fixed a minimal parabolic subgroup P_0 of G and a Levi subgroup M_0 of P_0; let T_0 be the maximal split torus of the centre of M_0 (T_0 does not depend on the choices made and in particular on the choice of P_0 except up to conjugation by an element of $G(k)$). We denote by $R(T_0, G)$ the set of roots of G relative to T_0. It forms a root system in the sense of [B] Chapter 6. Recall that with every root one can associate a coroot which is a one-parameter subgroup of T_0 ([BT] §7 reduces this to the split case).

There is a canonical duality with values in $\mathbb{Z}$ between the subgroup of rational characters of a split torus and that of its one-parameter subgroups, which we denote by $\langle\, ,\rangle$. For every $\chi\in \mathrm{Rat}(M_0)$ and every coroot $\check\beta$, this allows us to define $\langle\chi,\check\beta\rangle := \langle \mathrm{res}_0\chi,\check\beta\rangle$, where res_0 is the restriction of M_0 to T_0. This extends $\mathbb{R}$-linearly to define $\langle\lambda,\check\beta\rangle$ for all $\lambda\in \mathrm{Re}\, X_{M_0}\simeq \mathrm{Re}\,\mathfrak{a}^*_{M_0}$, and $\mathbb{C}$-linearly to define $\langle\lambda,\check\beta\rangle$ for all $\lambda\in\mathfrak{a}^*_{M_0}$. In particular, every coroot can be identified with an element of $\mathrm{Re}\,\mathfrak{a}_{M_0}$. If k is a number field, this defines $\langle\lambda,\check\beta\rangle$ for all $\lambda\in X_{M_0}$. If k is a function field, $\langle\lambda,\check\beta\rangle$ cannot be defined for $\lambda\in X_{M_0}$. See, however, I.1.8 below.