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112 Schur algebras and representation theory

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To the clan Martin

In my books I am solemn, sweet, refined; in real life I am rather vehement, sharp and contemptuous, a busy mocker. But I am also something of a fatalist. However, I am going to try to leave the Long free for writing and to have a subject ready to begin upon....I think I ought to be able to write rather a good story—if I weren't really so *lazy*: that is my main trouble, my hurried exhuberance.

A. C. Benson (1862–1925), *Diary, December 21, 1913*

Contents

Introduction

Between the years 1896 and 1900, G. Frobenius [1896], [1900] invented and mapped out the theory of complex representations of finite groups, paying special attention to representations of the symmetric group Σ_r. Thus, given a finite group we want to determine all group homomorphisms from our group into $\mathrm{GL}_m(\mathbf{C})$ for arbitrary positive integers m. Around this time Frobenius suggested to his pupil Issai Schur that he might examine the representation theory of the infinite group $\Gamma_{\mathbf{C}} = \mathrm{GL}_n(\mathbf{C})$. The subsequent investigations appeared in Schur's [1901] beautiful doctoral thesis of 1901. Schur studied $\Gamma_{\mathbf{C}}$ by means of the $\mathbf{C}$-space $A_{\mathbf{C}}(n, r)$ of r-homogeneous polynomial functions in the n^2 coordinate functions on $\Gamma_{\mathbf{C}}$. In particular he showed that the isomorphism types of irreducible representations of $\mathbf{C}\Gamma_{\mathbf{C}}$ with a given degree of homogeneity r are in one-to-one correspondence with the partitions of r into at most n parts; he also showed that the character of an irreducible module indexed by such a partition λ is a certain symmetric function, now called a 'Schur function', s_λ. In order to exploit Frobenius' work on Σ_r he set up (in modern parlance) an equivalence between the category of polynomial representations and the module category of Σ_r.

In another paper, Schur [1927] took a rather different approach. He analysed the actions of both Σ_r and $\Gamma_{\mathbf{C}}$ on the rth tensor power of the natural module for $\Gamma_{\mathbf{C}}$ (the space of n-columns over $\mathbf{C}$). Tensor space is then a permutation module and, using this instead of the function space, he thereby obtained shorter and very elegant proofs of results in his thesis. The procedure received publicity in Weyl's great treatise [1973], first published in 1939.

It is natural to present a characteristic-free treatment of general linear and symmetric groups. The subject began with the work of Thrall [1942] and led to the fundamental paper of Carter and Lusztig [1974].

They took up the representation theory of $\Gamma_K = \mathrm{GL}_n(K)$ over an infinite field, K, of arbitrary characteristic. This time the main tool is the hyperalgebra $\mathcal{U}_K$ constructed out of the Kostant $\mathbf{Z}$-form of the universal enveloping algebra of the general linear Lie algebra over $\mathbf{Q}$. They produce explicit polynomial generators for the centre of $\mathcal{U}_{\mathbf{Q}}$, with nice multiplicative properties, and these are used to produce the required generalisations of Schur's classical results. In particular they constructed the 'Weyl modules' as certain subspaces of tensor space, showed they were defined over $\mathbf{Z}$ and hence could be regarded as reductions modulo p of the modules studied by Schur in characteristic zero. Another approach was pioneered by Clausen [1979, 1980] whose main tool was the letter place algebra.

Green's monograph appeared in 1980 (Green [1980]). He placed these developments in the context of certain finite-dimensional K-algebras, which he christened Schur algebras, and clarified the connections between representations of Γ_K and the symmetric group. In this scenario the Schur algebra is the dual of the 'Schur coalgebra' $A_K(n,r)$ of r-homogeneous polynomial functions on Γ_K, and may be identified with the centralising algebra of the Σ_r-action on r-tensor space of the natural Γ_K-module. The influence of these hundred or so pages is hard to over-emphasise: throughout the eighties most work on Schur algebras was inspired by this presentation. His basic approach was combinatorial, with no overt appeal to algebraic group theory, and as such is in the same spirit as Schur's original exposition.

My intention in this book is to expand on Green's treatment of Schur algebras, and also to write a fairly full account of the exciting developments which have occurred in the time since Green's work was first published. I would like to mention a selection of these developments straight away. The work of Cline, Parshall and Scott, together with results of Ringel [1991], on highest weight categories (the module categories for so-called quasi-hereditary algebras) has been very influential in recent years. These ideas were arrived at by abstraction from the classical representation theory of semisimple algebraic groups and Lie algebras. In some sense the abstract versions are 'characteristic-free,' the unifying theme being an axiom which postulates the existence of certain filtrations which are the natural analogue of composition series in characteristic zero. Parshall [1989] proved in that the Schur algebra, $S_K(n,r)$, is quasi-hereditary, hence its module category is a highest weight category. In a series of papers, Green [1990b], [1991a], [1991b] and [1992] gave elegant constructions of the costandard modules (called

Schur modules in this text) and the standard modules (called here the Weyl modules) for this category. Meanwhile Donkin [1981] had also proved that the module category for $S_K(n,r)$ was a highest weight category, and in a remarkable set of papers (Donkin [1986], [1987], [1992], [1993]) he uses the machinery of algebraic group theory to prove deep theorems about the modular representation theory and associated block theory of the Schur algebras. The interplay between the block theory of symmetric groups and the Schur algebras has received quite a lot of attention recently. A study of Schur algebras of finite representation-type was initiated by Erdmann [1993a], and has now been developed by various people in several directions, see Erdmann, Martin and Scopes [1993], Erdmann [1993b] and Donkin and Reiten [1993].

Donkin was one of several authors to consider the cohomology theory of the Schur algebra; in particular he give a direct proof that the Schur algebras have finite global dimension (before the birth of quasi-hereditariness), which at the time was a great surprise. Such characteristic-free themes have also been employed by Akin *et al.*, (see, for example, Akin and Buchsbaum [1985], [1988]) in constructing explicit projective resolutions of Weyl modules.

Finally, a long collection of papers by Dipper and James [1986], [1987], [1989], [1991] is devoted to the study of Hecke algebras of type A and the action of them on tensor space; in the process, they introduce the q-Schur algebras, the centralising algebra of this action. The module category of this algebra is equivalent to the category of polynomial representations of a certain quantum group; putting $q = 1$ we re-obtain the classical situation. But more is true: if q is a prime power and coprime to p we obtain strong information on the p-modular representations of the finite reductive group $\mathrm{GL}_n(q)$. Indeed the calculation of decomposition numbers for the various structures mentioned above might be considered the central theme of the whole text.

My presentation is designed as a shop window for these new ideas: I hope that the fairly comprehensive bibliography at the end will aid the reader wanting to go into a topic more deeply. Let me now advertise the forthcoming attractions chapter by chapter.

Chapter 1 opens with a précis of Schur's thesis topic. This conveniently allows us to define frequently used notation and explain the underlying combinatorics necessary for passage to finite characteristic. In Theorem 1.6.1 we show that the isomorphism types of the simple r-homogeneous polynomial modules are in bijective correspondence with the set $\Lambda^+(n,r)$ of 'dominant weights', i.e. the set of partitions of r

into at most n parts. The proof requires little beyond familiarity with symmetric functions and characters.

These preliminaries complete, in Chapter 2 we introduce the Schur algebra, $S_K(n,r)$. We characterise this finite-dimensional K-algebra as the centralising algebra of the right action of the symmetric group Σ_r on r-tensor space $E^{\otimes r}$, where E is the natural module of n-columns for Γ_K. In Theorem 2.2.7 we give the important result that the category $\mathcal{P}_K(n,r)$ of r-homogeneous polynomial representations of Γ_K is equivalent to the module category for the Schur algebra. The second part of this chapter is devoted to proving that certain elements of the coalgebra $A_K(n,r)$, called standard bideterminants, form a basis of $A_K(n,r)$. The basic combinatorial input is the famous Straightening Formula, an old result in multilinear algebra with a long history extending back to ideas of Young, and indeed beyond into the mists of time to the founders of invariant theory. Since $A_K(n,r)$ is the K-dual of the Schur algebra, so we find that in the process a basis of 'standard codeterminants' is obtained for $S_K(n,r)$.

In Chapter 3 the search begins in earnest for the simple objects of $\mathcal{P}_K(n,r)$. We show that given any $\lambda \in \Lambda^+(n,r)$ there exist $S_K(n,r)$-modules $V(\lambda)$ and $M(\lambda)$, which I shall call Weyl modules and Schur modules respectively, both equipped with bases of bideterminants or codeterminants. In section 3.3 we discuss in some detail the fundamental notion of quasi-hereditary algebras and prove that Schur algebras have this property by exhibiting an explicit defining chain of ideals. It is seen that the Schur modules and Weyl modules mentioned above are precisely the costandard and standard modules in the highest weight category $\mathcal{P}_K(n,r)$. An easy consequence is the conclusion that Schur algebras have finite global dimension. The cohomology of Schur algebras deserves a book to itself, and all we do here is summarise some of the recent endeavours towards the construction of explicit resolutions and other aspects of cohomology theory.

Chapter 4 brings symmetric groups into the fray. If one assumes that $r \leq n$ then existence of a weight $\omega = (1^r)$ is guaranteed. In turn, the idempotent $e = \xi_\omega \in S_K(n,r)$ defined by this weight induces a functor f_e between the category $\mathcal{P}_K(n,r)$ and the module category for $K\Sigma_r$ (map a module in the first category to its ω-weight space). It is precisely this functor that gave Schur a category equivalence over $\mathbf{C}$, but there are complications in finite characteristic; for example, complete reducibility does not carry over. By analysing the images of Schur modules and the Weyl modules under f_e, one can construct the Specht modules for

$\mathbf{Q}\Sigma_r$ and the p-modular irreducibles. The Schur functor is used to outline some interesting conjectures about modular irreducibles for both symmetric groups and general linear groups, and also it features in the treatment of James' theorem in section 4.4: for $r \leq n$, this exhibits the decomposition matrix for Σ_r as a submatrix of that for Γ_K. The final two sections define and study Young modules. These are certain trivial source modules for Σ_r, and a knowledge of how $E^{\otimes r}$ decomposes into Young module components is seen as equivalent to a knowledge of Γ_K decomposition numbers. Use is made of the existence of certain filtrations whose sections are all Schur modules—such '∇-good filtrations' are a by-product of the representation theory of the quasi-hereditary algebra $S_K(n,r)$.

Throughout, a common theme is to proceed by analogy with Brauer's theory of modular representations of finite groups and, as noted above, we see in Chapter 4 that one has a concept of modular reduction for Γ_K and hence p-modular decomposition numbers. Another important aspect of Brauer's theory hinges on the notion of a p-block. Donkin [1980] investigated block decompositions for the reductive group Γ_K and the semisimple simply connected algebraic groups like SL_n, where the affine Weyl group determines the blocks. One might expect that the Schur algebras have similar properties, and indeed this is the case. A rule akin to the famous 'Nakayama Conjecture' for Σ_r has been proved by Donkin [1992] to give the blocks of $S_K(n,r)$. In general this amounts to stripping off hooks of p-power length, instead of the more familiar case of Σ_r, where one strips off p-hooks. As already suggested, this proof involves some algebraic group theory, the basics of which appear in summary in the Appendix. This done, we prove a finiteness theorem by showing that in fact there are only finitely many Morita equivalence classes of blocks of $S_K(n,r)$ for given n, r: this is an analogue of a result for G, due to Scopes [1991]. Some examples of blocks of $S_K(n,r)$ of finite type are given.

The last two chapters are designed as an introduction to the theory of q-Schur algebra. I have settled on an informal survey of this very active area of research. We begin by defining a q-deformation of the coordinate ring of $n \times n$ matrices to obtain a graded bialgebra; on taking its rth homogeneous component there is produced a coalgebra $A_q(n,r)$, whose dual is defined to be the q-Schur algebra. Specialising to $q = 1$ we obtain the usual Schur algebra $S_K(n,r) = A_K(n,r)^*$. At present there are several versions of the q-Schur algebra in the literature (Dipper and James [1989], Dipper and Donkin [1991], Parshall and Wang [1991]) and we spend most of Chapter 6 showing that the definitions are equivalent. We will identify $S_q(n,r)$ and $\mathrm{End}_{\mathcal{H}_q}(\oplus M_q^\lambda)$, where $\mathcal{H}_q$ is a certain Hecke algebra reducing to $K\Sigma_r$ if $q=1$, and M_q^λ $(\lambda \in \Lambda^+(n,r))$ is a q-version of the well-known permutation module M^λ on cosets of Young subgroup Σ_λ. The direct sum of the M_q^λ over all compositions of r is a version of tensor space, and $\mathcal{H}_q$ acts on this space as described in section 6.6; factoring out the kernel gives the q-Schur algebra as originally defined; see Theorem 6.6.7.

We produce a q-version of the coordinate ring $K[\Gamma]$ of Γ_K; there emerges an algebra, $K[\Gamma_q]$, which it is convenient to think of as the coordinate ring of a 'quantum group' Γ_q over K. It is shown that the category of polynomial $K[\Gamma_q]$-modules is equivalent to the module category for $S_q(n,r)$, and since $S_q(n,r)$ is quasi-hereditary, this module category is a highest weight category. Thus there are costandard modules (the q-Schur modules) and standard modules (the q-Weyl modules). The study of these modules we adopt is combinatorial, with little use of the now extensive theory of quantum groups developed by Manin [1988], Parshall and Wang [1991] and Andersen *et al.* [1991]. The q-Schur algebras $S_q(n,r)$ originally were studied by Dipper and James [1989] in order to link up the p-modular representations of Σ_n, representations of $\mathrm{GL}_n(K)$ and representations of $\mathrm{GL}_n(q)$ over fields whose characteristic is coprime to q. Once again there are q-Young modules, and a knowledge of their multiplicities as summands of M_q^λ is equivalent to a knowledge of the decomposition numbers of $S_q(n,r)$, i.e. of the multiplicities of the simple modules in the q-Weyl modules: a positive solution to this problem would in turn give the decomposition numbers for all the structures just mentioned.

Notation and Conventions. The following notation is fixed throughout.

- We will use **N**, $\mathbf{N}_0$, **Z**, **Q**, **R** and **C** for the sets of natural numbers, non-negative integers, integers, rationals, reals and complex numbers, respectively.
- $X \subseteq Y$ means that X is a subset of Y, $X = \emptyset$ or Y being included.
- $V \oplus W$ stands for the direct sum of the vector spaces V and W; there are obvious extensions to finite and arbitrary direct sums.
- $\langle e_1, \ldots, e_n \rangle$ stands for the vector space spanned by $e_1, \ldots, e_n$.
- $\square$ will signify the end of a proof.

Lists of all other commonly used notation appear in indexes at the end of the book. The numbering inside the chapters is as follows: equations referred to in the text are labelled by bracketed numbers, thus 'as seen in (4.6)' refers to equation (4.6) contained within Chapter 4. Sections, lemmas and theorems etc. are identified by unbracketed numbers, thus 'as proved in 4.6 and see also 4.6.1' refers to section 4.6 in Chapter 4, and to a result labelled 4.6.1 contained therein. The exception to this schema is that Section 4 in the Appendix will be referred to as A.4.

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