# 1

# Polynomial functions and combinatorics

## 1.1 Introductory remarks

This chapter opens with a brief summary of Schur's work on the complex polynomial representations of the general linear group $\mathrm{GL}_n(\mathbf{C})$ of $n \times n$ nonsingular matrices with entries in $\mathbf{C}$; we then proceed to analyse ways of extending his ideas to representations over an infinite field $K$ of arbitrary characteristic. Thus, Schur considered a polynomial matrix representation of $M \in \mathrm{GL}_n(\mathbf{C})$, i.e. a group homomorphism $T : \mathrm{GL}_n(\mathbf{C}) \to \mathrm{GL}_m(\mathbf{C})$ such that each entry of $T(M)$ could be expressed as a polynomial with complex coefficients in the entries of $M$; however we shall study a $K\mathrm{GL}_n(K)$-module $V$ whose coefficient space lies in the bialgebra $A_K(n)$ generated by the $n^2$ coordinate functions. In this way $V$ affords a matrix representation whose entries are polynomials over $K$ in the original entries. We shall then take the $r$th homogeneous component, $A_K(n,r)$, of $A_K(n)$ and consider the category of all representations having coefficient space lying in this component. Basic properties of the coalgebra $A_K(n,r)$ are catalogued in 1.3.

Throughout the text the methods employed use a blend of the standard combinatorial machinery. Nowadays I can safely assume that the combinatorics of partitions and their associated Young diagrams is well known, so no more than a cursory treatment is given in 1.4. This leads to one definition of a weight (as a certain $n$-tuple of integers) and we connect this idea with the algebraic group concept of weight (arising from the action of the torus) in 1.5. At the same time we provide a short discussion on formal and natural characters of representations of $\mathrm{GL}_n(K)$ and compute these for a few accessible but important examples. Technical matters relating to the representation theory of algebraic groups are safely relegated to the Appendix. Notwithstanding some simple lemmas on weight spaces in 1.6, we already have sufficient material to show that the isomorphism classes of the irreducible objects in the category ${}_{A_K(n,r)}\textbf{mod}$ of finite-dimensional left $A_K(n,r)$-modules are in one-to-one correspondence with partitions of $r$ into at most $n$ parts. Concrete constructions of these modules are made in the next two chapters.

## 1.2 Schur's thesis

We choose $2n^2$ independent variables $a_{\alpha\beta}$, $b_{\alpha\beta}$, where $\alpha$ and $\beta$ run independently through the set $\mathbf{n} = \{1, 2, \ldots, n\}$; we may regard these as entries in the two $n \times n$ matrices $A = [a_{\alpha\beta}]$, $B = [b_{\alpha\beta}]$. Let $T = T(A) = [t_{\gamma\delta}]$ be some $m \times m$ matrix ($m \in \mathbf{N}$) with entries $t_{\gamma\delta}$ belonging to the $\mathbf{C}$-algebra $\mathbf{C}[a]$, generated by the $a_{\alpha\beta}$ ($\alpha$, $\beta \in \mathbf{n}$). We observe therefore that each $t_{\gamma\delta}$ is a polynomial with complex coefficients in the $n^2$ entries $a_{\alpha\beta}$ of the matrix $A$.

Suppose that $C = AB$, that is $C = [c_{\alpha\beta}]$ with

$$c_{\alpha\beta} = \sum_{\mu \in \mathbf{n}} a_{\alpha\mu} b_{\mu\beta}. \tag{1.1}$$

Thus the entries of $C$ lie in the polynomial algebra $\mathbf{C}[a, b]$ generated by the $a_{\alpha\beta}$ and $b_{\alpha\beta}$. Now Schur was interested in the case where $T$ is *invariant*, that is, $T$ has the following property:

$$T(A)\,T(B) = T(C).$$

Simple examples are $T(A) = A$ with $m = n$, or if $m = 1$ we might take $T(A) = [\det A]$.

If $g, h \in \Gamma_{\mathbf{C}} = \mathrm{GL}_n(\mathbf{C})$, then by (1.1) we have $T(g)T(h) = T(gh)$, and so $T : \Gamma_{\mathbf{C}} \to \mathrm{GL}_m(\mathbf{C})$ is a matrix representation of $\Gamma_{\mathbf{C}}$ of dimension $m$, provided we assume additionally that $T(I_n) = I_m$. We shall call $T$ a *complex polynomial representation* of $\Gamma_{\mathbf{C}}$ of degree $m$.

**Remark.** Actually, Schur started by considering *rational representations*, i.e. those $T(A)$ whose entries were rational functions of the entries of the matrix $A$. In this case we may write $T(A) = p(A)q(A)^{-1}T'(A)$, where $A \mapsto p(A)$, $A \mapsto q(A)$ are maps into $\mathbf{C}[a]$, with $p$ and $q$ polynomial in the entries of $A$, and $T'(A)$ is polynomial as above. Hence it is no loss to consider those representations $T$, as mentioned above, whose entries lie in the ring $A_{\mathbf{C}}(n)$ of all polynomial functions $\mathrm{GL}_n(\mathbf{C}) \to \mathbf{C}$.

**Definition 1.2.1** The invariant matrices $T$ and $U$ of degree $m$ are *equiv-*

*alent* if there exists $P \in \mathrm{GL}_m(\mathbf{C})$ such that $P^{-1}TP = U$. We write $T \simeq U$ in this case. We say $T$ is *reducible* (over $\mathbf{C}$) if

$$T \simeq \begin{bmatrix} T_1 & V \\ 0 & T_2 \end{bmatrix},$$

where $T_i$ is an invariant matrix of degree $m_i$ and $V$ is some $m_1 \times m_2$ matrix with entries in $A_{\mathbf{C}}(n)$. If $T$ cannot be written in this way it is termed *irreducible*. Finally, if $T$ is equivalent to a matrix in block diagonal form in which each diagonal block matrix is irreducible, then $T$ is said to be *completely reducible*. These definitions also apply to matrix representations of groups over arbitrary fields.

Suppose that $T(A)$ is invariant, and $x$ is some variable independent of the $a_{\alpha\beta}$. Then $T(xA) = T_0(A) + xT_1(A) + \cdots + x^r T_r(A)$ for some $r \in \mathbf{N}$. Each $T_i(A)$ is itself invariant. Further, $T$ is equivalent to the block diagonal matrix $\mathrm{diag}(T_0, \ldots, T_r)$. To see this, let $E_i = T_i(I_n)$. Then $T(xI_n) = E_0 + xE_1 + \ldots + x^r E_r$ for all $x \in \mathbf{C}$. The invariance condition $T(xI_n)T(A) = T(xA) = T(A)T(xI_n)$ forces

$$\begin{aligned}(E_0 + xE_1 + \cdots + x^r E_r)(T_0(A) + T_1(A) + \cdots + T_r(A)) = \\ T_0(A) + xT_1(A) + \cdots + x^r T_r(A)\end{aligned}$$

for all $x \in \mathbf{C}$. Equating coefficients implies $E_i T_i(A) = T_i(A)E_i$. Now from linear algebra we may find $P \in \mathrm{GL}_m(\mathbf{C})$ such that, for each $i$, $P^{-1}E_iP = E_i'$, where $E_i'$ has $m_i$ consecutive 1s down the diagonal and 0s everywhere else, such that the sum of the $E_i'$ equals $I_n$. Here $m_i$ is the order of $T_i$ (so the sum of the $m_i$ is the order of $T$). Writing $P^{-1}T_i(A)P = T_i'(A)$ we obtain the relations $E_i'T_i'(A) = T_i'(A)E_i'$ for all $i$. Hence $T_i'(A)$ has zero entries everywhere except in those rows and columns in which $E_i'$ has 1s. We conclude that $P^{-1}T(A)P$ is the diagonal sum of the matrix blocks $T_i'(A)$ as required.

Now the entries in each $T_p$ are homogeneous polynomials of the same degree $p$ in the entries of $A$, that is, $T_p(xA) = x^p\, T_p(A)$. Thus we lose nothing in assuming henceforth that $T$ is already homogeneous of some degree, say $r$. Equivalently, we may assume that each entry $t_{\gamma\delta}$ of $T$ lies in $A_{\mathbf{C}}(n,r)$, the subspace of $A_{\mathbf{C}}(n)$ generated by functions that are homogeneous of total degree $r$ in the $n^2$ variables $a_{\alpha\beta}$.

### Notation

Before we can describe a basis for $A_{\mathbf{C}}(n,r)$, we list some notation, which, in fact, will hold throughout the text.

- If $K$ is an arbitrary infinite field and $S$ is a $K$-algebra, we write ${}_S\mathbf{mod}$ for the category of all finite-dimensional left $S$-modules; $\mathbf{mod}_S$ will be the category of finite-dimensional right $S$-modules. If $V, W \in {}_K\mathbf{mod}$ we usually write $V \otimes W$ for $V \otimes_K W$.
- If $H$ is a finite group, we denote its group algebra by $KH$.
- $I = I_n = I(n,r) = \{\text{maps from } \mathbf{r} \text{ to } \mathbf{n}\} = \{i = (i_1, \ldots, i_r) : i_\rho \in \mathbf{n}, \ \forall \rho \in \mathbf{r}\}$ is the set of *multi-indices* (which some authors write in the power notation $\mathbf{n}^{\mathbf{r}}$).
- $G = \Sigma_r$, the symmetric group on $\mathbf{r}$; more generally, if $X \subseteq \mathbf{r}$, we write $\Sigma_X$ for the subgroup of $G$ fixing every element outside $X$. $G$ acts naturally on the right on $I$ by place permutations:

  $$i\pi = (i_{\pi(1)}, \ldots, i_{\pi(r)}) \qquad (i \in I, \ \ \pi \in G).$$

  $G$ also acts on $I \times I$ as follows:

  $$(i,j)\pi = (i\pi, j\pi) \quad (i \in I, \ j \in I, \ \ \pi \in G).$$

- We write $i \sim j$ when $i, \ j \in I$ are in the same $G$-orbit. Similarly, we write $(i,j) \sim (p,q)$ when $(i,j),(p,q) \in I \times I$ are in the same $G$-orbit, i.e. if $p = i\pi$ and $q = j\pi$ for some $\pi \in G$.
- If $i \in I$ we write $G_i = \{\pi \in G : \ i\pi = i\}$ for the stabiliser of $i$.

Evidently, $A_{\mathbf{C}}(n,r)$ has a $\mathbf{C}$-basis consisting of all monomials of degree $r$:

$$a_{i,j} = a_{i_1 j_1} \cdots a_{i_r j_r} \quad (i_\rho, \ j_\rho \in \mathbf{n} \ \ \forall \rho \in \mathbf{r}). \tag{1.2}$$

Notice our convention in writing $a_{i,j}$ (with a comma) when the subscripts are multi-indices, but $a_{pq}$ (without a comma) if the subscripts are integers. A given monomial may have two different expressions of the form (1.2). Indeed for $i, j, i', j' \in I$,

$$a_{i,j} = a_{i',j'} \Leftrightarrow (i,j) \sim (i',j').$$

We now have a basis for $A_{\mathbf{C}}(n,r)$:

**Theorem 1.2.2** *Choose arbitrarily a set $\Omega$ of representatives of the $G$-orbits of $I \times I$. Then*

$$\{a_{i,j} : (i,j) \in \Omega\}$$

*is an irredundantly described basis of* $A_{\mathbf{C}}(n,r)$. *Hence*

$$\dim A_{\mathbf{C}}(n,r) = \binom{n^2+r-1}{r}.$$

*Proof* The number of distinct monomials of degree $r$ in the $n^2$ variables $a_{\alpha\beta}$ is easily seen to be given by the quoted binomial coefficient. □

With the above notation established, we return to the general problem of finding conditions on the $m \times m$ matrix $T(A)$ that make it invariant. The entries of $T(A)$ lie in $A_{\mathbf{C}}(n,r)$, so using the basis provided by 1.2.2 we may write

$$T(A) = \sum_{(i,j)\in\Omega} a_{i,j} M_{i,j}$$

where each $M_{i,j}$ is some $m \times m$ matrix with complex entries. Thus

$$T(A)T(B) = \sum_{(i,j)\in\Omega} \sum_{(k,l)\in\Omega} a_{i,j} b_{k,l} M_{i,j} M_{k,l}.$$

By the definition of invariance this must be equal to

$$T(C) = \sum_{(p,q)\in\Omega} c_{p,q} M_{p,q},$$

where, from (1.1), the coefficients satisfy

$$c_{p,q} = c_{p_1 q_1} \dots c_{p_r q_r} = \prod_{\rho} \sum_{h_\rho \in \mathbf{n}} a_{p_\rho h_\rho} b_{h_\rho q_\rho} = \sum_{h \in I} a_{p,h} b_{h,q}.$$

Certainly $(p,q) \in \Omega$, but there is no reason why either of the pairs $(p,h)$ and $(h,q)$ have to lie in $\Omega$. We may write the last formula as

$$c_{p,q} = \sum_{(i,j)\in\Omega} \sum_{(k,l)\in\Omega} \gamma(i,j,k,l,p,q) a_{i,j} b_{k,l},$$

where, for any $(p,q)$, $(i,j)$, $(k,l) \in \Omega$,

$$\gamma(i,j,k,l,p,q) = |\{h \in I : (p,h) \sim (i,j),\ (h,q) \sim (k,l)\}|.$$

By comparing coefficients, we conclude that $T(A)T(B) = T(C)$ if and only if

$$M_{i,j}M_{k,l} = \sum_{(p,q)\in\Omega} \gamma(i,j,k,l,p,q)M_{p,q}$$

for all $(i,j)$, $(k,l) \in \Omega$. Experts will recognise this somewhat unwieldy expression as Schur's famous 'Product Rule' .

It is hoped that the reader will now find it quite natural to define a linear $\mathbf{C}$-algebra $S_{\mathbf{C}}(n,r)$ (cf. Schur [1901, p. 29]) having as basis the symbols $\xi_{i,j}$, $(i,j) \in \Omega$, and multiplication defined by

$$\xi_{i,j}\xi_{k,l} = \sum_{(p,q)\in\Omega} \gamma(i,j,k,l,p,q)\xi_{p,q} \tag{1.3}$$

for all $(i,j)$, $(k,l) \in \Omega$.

**Definition 1.2.3** $S = S_{\mathbf{C}}(n,r)$ is the *complex Schur algebra* for $r$ and $n$.

By 1.2.2,

$$\dim_{\mathbf{C}} S = \binom{n^2+r-1}{r}.$$

Visibly, there is a one-to-one correspondence $R \longleftrightarrow R_0$ between

$$\left\{ \begin{array}{c} \text{invariant matrices} \\ T(A) = \sum a_{i,j}M_{i,j} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{matrix representations of } S \\ \xi_{i,j} \mapsto M_{i,j} \end{array} \right\}$$

respecting the representation-theoretic properties of equivalence and irreducibility. Using this principle, Schur showed that if $n \geq r$, there are further correspondences $R_0 \leftrightarrow R_1$ between the representations $R_0$ of $S$ and representations $R_1$ of $\Sigma_r$. The complex representations of $\Sigma_r$, determined by Frobenius [1896], [1900], together with the correspondences $R_0 \leftrightarrow R \leftrightarrow R_1$, were used by Schur to show that every invariant matrix was completely reducible. In modern terminology, the main problem he posed and solved was the following.

Given $n$, find a complete set of irreducible polynomial representations of degree $r$ for $\mathrm{GL}_n(\mathbf{C})$.

He showed that the set of such matrix representations was in one-to-one correspondence with the set $\Lambda^+(n,r)$ of partitions of $r$ into at most $n$ parts. Further, given some such matrix, say $T = T_\lambda(A)$ $(\lambda \in \Lambda^+(n,r))$, its trace, that is the character of the corresponding representation (which Schur termed the *characteristic*) was shown by an 'irrelevance argument' to be a certain symmetric function $\zeta_1, \ldots, \zeta_n$ of the eigenvalues of $A$. Using the above correspondences, he ordered the partitions and determined the characters by induction on the ordering. The character corresponding to $\lambda = (\lambda_1, \ldots, \lambda_n)$ is known as the *Schur function of type* $\lambda$, as in Macdonald [1979, I§3]. It is defined by the following quotient of a Vandermonde determinant

$$s_\lambda(\zeta_1, \ldots, \zeta_n) = \frac{|\zeta_i^{\lambda_i+n-j}|}{|\zeta_i^{n-j}|}.$$

**Remarks.**
(1) For more on the rich history of the development of the subject, the survey article of Dieudonné [1981] is recommended. For comments on the links with invariant theory, see the notes in Weyl [1973] and the exposition of Crawley-Boevey [1990].
(2) Schur's irrelevance argument effectively amounts to restricting the representation $T$ of $\Gamma_{\mathbf{C}}$ to the diagonal subgroup of $\Gamma_{\mathbf{C}}$: see 1.5.5 below.

**Example.** To illustrate some of these ideas, let us consider the following simple situation. We take $n = r = 2$ and let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. There are two types of irreducible invariant matrices.
(a) If $\lambda = (2, 0)$, we have a 3-dimensional irreducible representation

$$T_{(2,0)}(A) = \begin{bmatrix} a_{11}^2 & a_{11}a_{12} & a_{12}^2 \\ 2a_{11}a_{21} & a_{11}a_{22} + a_{12}a_{21} & 2a_{12}a_{22} \\ a_{21}^2 & a_{21}a_{22} & a_{22}^2 \end{bmatrix}.$$

Putting $A =$diag$(\zeta_1, \zeta_2)$ in $T_{(2,0)}(A)$, we obtain its character

$$s_{(2,0)}(\zeta_1, \zeta_2) = \zeta_1^2 + \zeta_1\zeta_2 + \zeta_2^2.$$

(b) The other case is to take $\lambda = (1, 1)$, which gives

$$T_{(1,1)}(A) = [a_{11}a_{22} - a_{12}a_{21}]$$

(that is, the determinant of $A$). This is a 1-dimensional representation with character

$$s_{(1,1)}(\zeta_1, \zeta_2) = \zeta_1\zeta_2.$$

**Remarks.**
(1) These representations are afforded by special cases of the Schur modules constructed in Chapter 3.
(2) Most of the definitions and concepts outlined above carry over in an

obvious way if $\mathbf{C}$ is replaced by $K$, an infinite field of arbitrary characteristic. In fact, as we shall see in Chapter 2, the correspondence $R \leftrightarrow R_0$ between polynomial representations of $\mathrm{GL}_n(K)$ of degree $r$ and representations of the Schur algebra $S_K(n,r)$ is still valid; however, the correspondence $R_0 \leftrightarrow R_1$ with $\Sigma_r$ is not so precise, for example, complete reducibility fails. Nevertheless, one can progress quite far. The irreducible polynomial representations may be constructed and are still labelled by $\Lambda^+(n,r)$, but the characters are as yet unknown for general $n, r$. We return to this point several times in the sequel.

**Integral forms.** One of the cornerstones of R. Brauer's modular representation theory of finite groups is the concept of $p$-modular reduction. In the study of invariant matrices $T_\lambda$ we say that $T_\lambda$ is *integral* if all its entries are polynomials in the $a_{\alpha\beta}$ in which the coefficients of the polynomials are integers. Hence we can interpret them as representations of $\mathrm{GL}_n(K)$ over any field $K$, or indeed over any commutative domain $R$. As examples both $T_{(2,0)}$ and $T_{(1,1)}$ of the last example are integral. Actually, thinking of these as representations of the Schur algebra $S$ via the correspondence $R \leftrightarrow R_0$, it is a splendid fact that one can always find matrix representations for $S$ which are integral. These matters were first investigated by Carter and Lusztig [1974, 3.5] who gave explicit integral forms for the Weyl modules (the duals of the Schur modules referred to above). See also Clausen [1979] and Akin and Buchsbaum [1985] for other aspects of the problem. As usual there are many possible choices for a $\mathbf{Z}$-form:

**Example.** Returning to the above example, the representation

$$T'_{(2,0)}(A) = \begin{bmatrix} a_{11}^2 & 2a_{11}a_{12} & a_{12}^2 \\ a_{11}a_{21} & a_{11}a_{22} + a_{12}a_{21} & a_{12}a_{22} \\ a_{21}^2 & 2a_{21}a_{22} & a_{22}^2 \end{bmatrix}$$

is integral and is equivalent over $\mathbf{C}$ to $T_{(2,0)}$, but as representations of $\mathrm{GL}_2(K)$ $T_{(2,0)}$ and $T'_{(2,0)}$ are not equivalent over fields of characteristic 2—they are not even irreducible!

This is not a serious drawback: we are usually interested only in the composition factors of these representations. These turn out to be the same, whichever $\mathbf{Z}$-form is chosen, by a theorem of Brauer and Nesbitt (see Curtis and Reiner [1981, (16.16)]).

## 1.3 The polynomial algebra

As a first step in generalising Schur's procedure, discussed in the last section, we replace $\mathbf{C}$ by an arbitrary infinite field $K$. Consider the set $K^\Gamma$ of all maps from $\Gamma = \Gamma_K = \mathrm{GL}_n(K)$ to $K$. It is clear that $K^\Gamma$ is a commutative $K$-algebra having addition and multiplication defined pointwise, and with identity $1 : x \mapsto 1_K$. In fact, we have

**Lemma 1.3.1** *$K^\Gamma$ is a $(\Gamma, \Gamma)$-bimodule under the actions of left and right translation of functions; that is, for $f \in K^\Gamma$ and $g \in \Gamma$, define $g \circ f$ and $f \circ g$ in $K^\Gamma$ by*

$$\begin{aligned} (g \circ f)(x) &= f(xg) \\ (f \circ g)(x) &= f(gx) \end{aligned}$$

*for all $x \in \Gamma$. These actions are linear, multiplicative and the left and right actions commute.*

**Definition 1.3.2** For each pair $\alpha$, $\beta \in \mathbf{n}$, define the *coefficient functions* or *coordinate functions* $c_{\alpha\beta} \in K^\Gamma$ by

$$c_{\alpha\beta}(g) = g_{\alpha\beta}$$

for all $g = [g_{\alpha\beta}] \in \Gamma_K$. The *algebra of polynomial functions* on $\Gamma_K$, denoted $A_K(n)$, is the subalgebra of $K^\Gamma$ generated by the $n^2$ coefficient functions $c_{\alpha\beta}$ $(\alpha, \beta \in \mathbf{n})$.

The set $\{c_{\alpha\beta} : \alpha\ \beta \in \mathbf{n}\}$ is algebraically independent over $K$ since $K$ is infinite, and so $A_K(n)$ can be regarded as the algebra of all polynomials in the $n^2$ indeterminates $c_{\alpha\beta}$. The elements of $A_K(n)$ are called *polynomial functions* on $\Gamma_K$. We often drop the subscript $K$ when the field is understood.

**Definition 1.3.3** For any $r \geq 0$, denote by $A_r = A_K(n, r)$ the $K$-subspace of $A_K(n)$ generated by those elements in $A_K(n)$ that, considered as polynomials in the $c_{\alpha\beta}$ are homogeneous of total degree $r$.

Clearly we have a grading

$$A_K(n) = \bigoplus_{r \geq 0} A_K(n, r)$$

by homogeneous degree, where we define $A_K(n, 0) = K1$. The argument justifying 1.2.2 can be generalised in an obvious way to give

**Theorem 1.3.4** *$A_r$ has $K$-basis*

$$\{c_{i,j} = c_{i_1 j_1} \cdots c_{i_r j_r} : (i,j) \in \Omega\},$$

*where $\Omega$ is a transversal of the set of all $G$-orbits on $I \times I$; hence*

$$\dim A_r = \binom{n^2 + r - 1}{r}.$$

The next observation relates $A_r$ to $K^\Gamma$.

**Lemma 1.3.5** *$A_r$ is a $(\Gamma,\Gamma)$-sub-bimodule of $K^\Gamma$ for all $r > 0$. In fact, for any $i, j \in I$ and $g$, $h \in \Gamma_K$,*

$$\begin{aligned} g \circ c_{i,j} &= \sum_{t \in I} g_{t,j} c_{i,t} \\ c_{i,j} \circ h &= \sum_{t \in I} h_{i,s} c_{s,j} \end{aligned}$$

*where we write $g_{t,j} = g_{t_1 j_1} \cdots g_{t_r j_r}$ (with a similar definition for $h_{i,s}$). Further, $A_0$ is a $(\Gamma,\Gamma)$-bimodule since $g \circ 1 = 1 = 1 \circ h$.*

*Proof* For any $x, g \in \Gamma$ and $\alpha, \beta \in \mathbf{n}$ we have, using 1.3.1,

$$(g \circ c_{\alpha\beta})(x) = c_{\alpha\beta}(xg) = [xg]_{\alpha\beta} = \sum_{\gamma \in \mathbf{n}} x_{\alpha\gamma} g_{\gamma\beta}$$

and so $g \circ c_{\alpha\beta} = \sum_\gamma c_{\alpha\gamma} g_{\gamma\beta}$. Thus

$$\begin{aligned} g \circ c_{i,j} &= g \circ (c_{i_1 j_1} \cdots c_{i_r j_r}) \\ &= \prod_{\rho \in \mathbf{r}} (g \circ c_{i_\rho j_\rho}) \\ &= \prod_{\rho \in \mathbf{r}} \sum_{t_\rho \in \mathbf{n}} g_{t_\rho j_\rho} c_{i_\rho t_\rho} \\ &= \sum_{t \in I} g_{t,j} c_{i,t}. \end{aligned}$$

Similarly for the right action. □

We now extend the actions of $\Gamma$ to actions of the group algebra $K\Gamma$. (Recall that if $H$ is an infinite group, the group algebra $KH$ is the collection of all $\sum \alpha_g g$, where $\alpha_g = 0$ for all but a finite number of the $g$.) Now, taking $\kappa = \sum_\Gamma \alpha_g g \in K\Gamma$ and $f \in K^\Gamma$, we put