In this treatise, the authors present the general theory of orthogonal polynomials on the complex plane and several of its applications. The assumptions on the measure of orthogonality are general, the only restriction is that it has compact support on the complex plane. In the development of the theory the main emphasis is on asymptotic behavior and the distribution of zeros.

In the first two chapters exact upper and lower bounds are given for the orthonormal polynomials and for the location of their zeros. The next three chapters deal with regular nth-root asymptotic behavior, which plays a key role both in the theory and in its applications. Orthogonal polynomials with this behavior correspond to classical orthogonal polynomials in the general case, and many extremal properties of measures in mathematical analysis and approximation theory turn out to be equivalent to this type of regularity. Several easy-to-use criteria are presented for regular behavior.

The last chapter contains applications of the theory, including exact rates for convergence of rational interpolants, best rational approximants, and nondiagonal Padé approximants to Markov functions (Cauchy transforms of measures). The results are based on potential-theoretic methods, so both the methods and the results can be extended to extremal polynomials in norms other than L^2 norms. The Appendix contains a brief introduction to potential theory.

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Herbert Stahl and Vilmos Totik
Frontmatter
More information

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General Orthogonal Polynomials

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

1 Luis A. Santalo *Integral geometry and geometric probability*
2 George E. Andrews *The theory of partitions*
3 Robert J. McEliece *The theory of information and coding: A mathematical framework for communication*
4 Willard Miller, Jr. *Symmetry and separation of variables*
5 David Ruelle *Thermodynamic formalism: The mathematical structures of classical equilibrium statistical mechanics*
6 Henryk Minc *Permanents*
7 Fred S. Roberts *Measurement theory with applications to decisionmaking, utility, and the social services*
8 L. C. Biedenham and J. D. Louck *Angular momentum in quantum physics: Theory and application*
9 L. C. Biedenham and J. D. Louck *The Racah–Wigner algebra in quantum theory*
10 W. Dollard and Charles N. Friedman *Product integration with application to differential equations*
11 William B. Jones and W. J. Thron *Continued fractions: Analytic theory and applications*
12 Nathaniel F. G. Martin and James W. England *Mathematical theory of entropy*
13 George A. Baker, Jr and Peter R. Graves-Morris *Padé approximants, Part I: Basic theory*
14 George A. Baker, Jr and Peter R. Graves-Morris *Padé approximants, Part II: Extensions and applications*
15 E. C. Beltrametti and G. Cassinelli *The logic of quantum mechanics*
16 G. D. James and A. Kerber *The representation theory of the symmetric group*
17 M. Lothaire *Combinatorics on words*
18 H. O. Fattorini *The Cauchy problem*
19 G. G. Lorentz, K. Jetter, and S. D. Riemenschneider *Birkhoff interpolation*
20 Rudolf Lidl and Harald Niederreiter *Finite fields*
21 William T. Tutte *Graph theory*
22 Julio R. Bastida *Field extensions and Galois theory*
23 John R. Cannon *The one-dimensional heat equation*
24 Stan Wagon *The Banach–Tarski paradox*
25 Arto Salomaa *Computation and automata*
26 Neil White (ed) *Theory of matroids*
27 N. H. Bingham, C. M. Goldie, and J. L. Teugels *Regular variation*
28 P. P. Petrushev and V. A. Popov *Rational approximation of real functions*
29 Neil White (ed) *Combinatorial geometries*
30 M. Pohst and H. Zassenhaus *Algorithmic algebraic number theory*
31 J. Aczel and J. Dhombres *Functional equations containing several variables*
32 Marek Kuczma, Bogden Chozewski, and Roman Ger *Iterative functional equations*
33 R. V. Ambartzumian *Factorization calculus and geometric probability*
34 G. Gripenberg, S.-O. Londen, and O. Staffans *Volterra integral and functional equations*
35 George Gasper and Mizan Rahman *Basic hypergeometric series*
36 Erik Torgersen *Comparison of statistical experiments*
37 Arnold Neumaier *Interval methods for systems of equations*
38 N. Korneichuk *Exact constants in approximation theory*
39 Richard A. Brualdi and Herbert J. Ryser *Combinatorial matrix theory*
40 N. White (ed) *Matroid applications*
41 Shoichiro Sakai *Operator algebras in dynamical systems: The theory of unbounded derivations in C^*-algebras*
42 W. Hodges *Model theory*

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Herbert Stahl and Vilmos Totik
Frontmatter
More information

Contents

Preface

The theory of orthogonal polynomials can be divided into two loosely related parts. One of them is the formal, algebraic aspect of the theory, which has close connections with special functions, combinatorics, and algebra, and it is mainly devoted to concrete orthogonal systems or hierarchies of systems such as the Jacobi, Hahn, Askey–Wilson, ... polynomials.

The investigation of more general orthogonal polynomials with methods of mathematical analysis belongs to the other part of the theory. Here the central questions are the asymptotic behavior of the polynomials and their zeros, recovering the measure of orthogonality, and so forth. This part has applications to approximation processes such as polynomial and rational interpolation, Padé approximation, and best rational approximation, to Fourier expansions, quadrature processes, eigenvalue problems, and so forth.

Textbooks on orthogonal polynomials usually cover material from both parts of the theory but give different emphasis in accordance with individual preference. Only the classical book [Sz3] by Gábor Szegő aims at a treatment of the subject in an encyclopedic manner. The present book is exclusively devoted to the second part of the theory. The main emphasis is on the investigation of the asymptotic behavior of *general* orthogonal polynomials, but related questions as, for instance, the distribution of zeros are also taken into consideration. A whole chapter is devoted to applications of the results in other areas.

Until now most of the asymptotic theory of orthogonal polynomials has concentrated on orthogonal systems for which the measure of orthogonality is supported on the real line or on the unit circle. Even then it has usually been assumed that the measure of orthogonality is sufficiently thick on its support. The present work is devoted to orthogonal polynomials with respect to *general measures* μ. The only requirement on μ is that it has compact support in $\mathbb{C}$, that is, both the support $S(\mu)$ of μ and the "thickness" of μ can be arbitrary. Thus, we allow both the measure and its support to be "wild," each of which has its own reflection in the general theory.

For orthonormal polynomials $p_n(\mu; z)$, as for sequences of polynomials in general, there exists a hierarchy of types of asymptotic behavior. We mention here the most common ones, which are called *power* (or Szegő), *ratio*, and *nth-root* asymptotic behavior. Roughly speaking, these mean that the sequences

$$\left\{ \frac{p_n(\mu; z)}{\varphi(z)^n} \middle| n \in \mathbb{N} \right\}, \tag{P.1}$$

$$\left\{ \frac{p_{n+1}(\mu; z)}{p_n(\mu; z)} \middle| n \in \mathbb{N} \right\}, \tag{P.2}$$

and

$$\{ \sqrt[n]{p_n(\mu; z)} \mid n \in \mathbb{N} \}, \tag{P.3}$$

respectively, tend to a limit on a certain set of values $z \in \mathbb{C}$ as $n \to \infty$ (in (P.1) the function φ has to be appropriately chosen). It is easy to see that each type of asymptotics in the hierarchy (P.1) to (P.3) implies the next one. Consequently, the nth-root asymptotic behavior is the most general of the three types and it requires the weakest assumptions. At the same time it is sufficient for many applications, as, for instance, the convergence of polynomial (Chebyshev–Fourier) expansions based on the system $\{p_n(\mu; z) \mid n \in \mathbb{N}\}$, or the convergence of continued fractions or Padé approximants to Markov functions (for other applications see Chapter 6).

In this context the present work can be classified as a monograph on nth-root asymptotic behavior. Earlier research in this direction has been mainly due to P. P. Korovkin, J. Ullman, P. Erdős, G. Freud, P. Turán, and H. Widom, although the case of general support has barely been touched upon. It was especially J. Ullman who systematically studied different bounds and asymptotics on orthogonal polynomials with respect to arbitrary measures μ on $[-1, 1]$, and we owe a lot to his research and personally to him for initiating and keeping alive the subject.

The present monograph synthesizes and considerably extends earlier research concerning general orthogonal polynomials. A large part of it (cf. Chapters 3–6) contains new results very often without any precedence (cf. Chapter 5, Sections 3.2–3.4, 4.2–4.6). We have put special emphasis on examples illustrating that our results are sharp. It has also been important for us to illustrate the possible connection with and the applicability of our theory to other fields of mathematical analysis (see Chapter 6).

Our proofs use potential-theoretic considerations. The usefulness of logarithmic potentials in the general theory can be easily understood if we recall that the modulus of a polynomial is basically nothing else than the exponential of a discrete potential. We very rarely use explicitly the orthogonality property – most of our proofs are based on the L^2 minimality of the monic orthogonal polynomials. Hence, our method works for L^p-extremal polynomials as well.

The content of the different chapters is briefly as follows. In Chapter 1 we give sharp upper and lower bounds for orthonormal polynomials and their leading coefficients. Chapter 2 examines the location and asymptotic distribution of the zeros. An extremely important concept, *"regular (nth-root) asymptotic behavior"* (in symbols, $\mu \in \mathbf{Reg}$), is introduced and characterized in Chapter 3. Polynomials with this property are the natural analogue of classical orthogonal polynomials in the general case, and they have many applications and equivalent formulations in different subjects of approximation theory. To facilitate these applications one needs easy-to-use criteria for $\mu \in \mathbf{Reg}$, which are given in Chapter 4. In Chapter 5 a surprising phenomenon is investigated: The regularity is basically a local property. Finally, Chapter 6 contains several applications of $\mu \in \mathbf{Reg}$ of which we mention here only the results in Sections 6.1 and 6.2, where the classical connection between continued fractions and orthogonal polynomials is extended to rational interpolation and best rational approximation of Cauchy transforms of measures μ, and it is shown that $\mu \in \mathbf{Reg}$ is equivalent to an exact maximal rate of convergence for these rational interpolants or approximants.

In the Appendix we assemble those results from the theory of logarithmic potentials that are frequently used in the text. A list of symbols and an index helps the reader to keep track of notations.

We used formula numbers of the form $(a.b)$, where a denotes the section number within the actual chapter, and b the formula number within the section. If we want to refer to a formula in a different chapter, then we use the form $(c.a.b)$, where c denotes the chapter number and a and b are as above.

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Herbert Stahl and Vilmos Totik
Frontmatter
More information

Symbols

$\partial\Omega$	boundary of Ω
$\mathbb{C}$	complex plane
$\overline{\mathbb{C}}$	extended complex plane, Riemann sphere
$\text{cap}(S)$	logarithmic capacity of S
$\text{cap}(V, S)$	condenser (or Green) capacity of the condenser (V, S)
$\text{card}(S)$	cardinality of S
$C(K)$	space of continuous functions on K
c_μ	minimal-carrier capacity corresponding to μ
$\text{Co}(S)$	convex hull of S
$\deg(P)$, $\deg P$	degree of the polynomial P
$D_\rho(x)$, $\Delta_\rho(x)$	disk of radius ρ with center at x
δ_x	Dirac measure at x
$\delta_{n,m}$	Kronecker's symbol
$\text{diam}(S)$	diameter of S
$\text{dist}(V, S)$	distance of V and S
$d(\mu, \nu)$	distance in the metric of the weak* topology
$d(z, E)$	distance of z and E
$\Gamma(\mu)$	set of carriers of μ
$\Gamma_0(\mu)$	set of minimal carriers of μ
$g_B(z; y)$	Green function of B with pole at y
$g(\alpha, \Omega; z)$	Green potential of α in Ω
$g_\mu(z; \infty)$	minimal-carrier Green function
$\gamma_n(\mu)$	leading coefficient of $p_n(\mu; \cdot)$
$\text{Im}(z)$	imaginary part of z
$I(\mu)$	logarithmic energy of μ
$\text{Int}(K)$	interior of K
$I(S)$	smallest interval containing $S \subseteq \mathbb{R}$
$\lambda(E)$, $\text{meas}(E)$	Lebesgue measure of E

$L(\mathbf{A})$	set of limit points of the interpolation scheme $\mathbf{A}$
$\mathcal{M}_\mu$	set of weak* limits of the normalized counting measures on the zeros of $p_n(\mu;\cdot)$
$\mathcal{M}[0,1]$	space of probability measures on $[0,1]$
$[m/n](z)$	Padé approximants
$\mathrm{MAX}p(\nu;\cdot)$	set of maximum points of $p(\nu;\cdot)$
$\|\mu\|$	total-variation measure corresponding to μ
$\\|\mu\\|$	total variation of μ, total mass
μ	positive Borel measure of compact support in $\mathbb{C}$
$\mu \sim \nu$	carrier related measures
μ−qu.e.	μ−a.e. except for a set of zero capacity
$\nu \ll \mu$	absolute continuity of ν with respect to μ
$\mathbb{N}$	set of natural numbers
$\nu_n \xrightarrow{*} \nu$, $\nu_n \to \nu$	weak* convergence of $\{\nu_n\}$ to ν
ν_S	counting measure on the set S
$\nu_{p_n(\mu;\cdot)}$	counting measure on the zeros of $p_n(\mu;\cdot)$
(ν,μ)	inner product with logarithmic kernel
ω_K	equilibrium measure of K
ω_μ	minimal-carrier equilibrium measure associated with μ
Ω	$\mathbb{C}\setminus \mathrm{Pc}(S(\mu))$, outer domain of $S(\mu)$
$\mathrm{Pc}(S)$	polynomial convex hull of S
$p_n(\mu;z)$	orthonormal polynomials with respect to μ
$p(\nu;z)$	logarithmic potential of ν
qu.e.	quasi everywhere
Π_n	set of polynomials of degree at most n
Π_n^*	set of monic polynomials of degree at most n
$q_n(\mu;z)$	monic orthogonal polynomials with respect to μ
$\mathbb{R}$	set of real numbers
$\mathrm{Re}(z)$	real part of z
Reg	set of regular measures
$\mathcal{R}_{m,n} \equiv \mathcal{R}_{mn}$	set of rational functions of numerator and denominator degrees at most m and n
$\mathcal{R}_n^{\mathbb{R}}$	set of rational functions with real coefficients of numerator and denominator degrees at most n
$r_n(f,A_n;z)$	interpolating rational function to f
$r_n^* = r_n^*(f,V;z)$	best rational approximant to f on V
$S(\mu)$, $\mathrm{supp}(\mu)$	support of μ
"$\subseteq$"	inclusion except for a set of zero capacity
$Z(p_n)$	set of zeros of p_n