1

Upper and Lower Bounds

This chapter is devoted to sharp upper and lower bounds for orthonormal polynomials with respect to general weights. The two bounds are given in terms of Green functions related to the carriers of the measure in question. As a corollary, sharp bounds are obtained for the leading coefficients. All subsequent chapters use both the notations and the results from the present one.

The chapter is organized as follows: Section 1.1 contains the statement of the main results; in Section 1.2 we prove some potential-theoretic preliminaries needed in the proofs. The actual proofs of the upper and lower estimates are carried out in Section 1.3, and the proof of their sharpness is given in Section 1.4. Finally, in Section 1.5 we construct some examples that illustrate the results.

1.1 Statement of the Main Results

The main results in this section are lower and upper asymptotic bounds for the nth root of the orthonormal polynomials $p_n(\mu; z)$ as $n \to \infty$, as well as their unimprovability.

In what follows $\operatorname{cap}(S)$ denotes the (outer logarithmic) *capacity* of a bounded set $S \subseteq \mathbb{C}$, that is, $\operatorname{cap}(S) = \inf_U \operatorname{cap}(U)$, where the infimum extends over all open sets $U \supseteq S$ (see Chapter 11, Section 2 of [La] or Appendix I), and we say that a property holds *qu.e.* (*quasi everywhere*) on a set $S \subseteq \mathbb{C}$ if it holds on S with possible exceptions on a subset of capacity zero. The abbreviation "qu.e." will be also used for "quasi every." By $g_B(z; \infty)$ we denote the (generalized) *Green function* with logarithmic pole at infinity associated with a Borel set $B \subseteq \overline{\mathbb{C}}$ with bounded complement $\overline{\mathbb{C}} \setminus B$. For our investigation and especially for the definition of the minimal-carrier Green function given in (1.3), it is essential that the Green function is defined for Borel sets $B \subseteq \overline{\mathbb{C}}$, and not only for domains in $\overline{\mathbb{C}}$, as it is the case in most text books. More details about the definition of Green functions will be given at the beginning of the next section and in the Appendix (see Appendix V).

μ will always denote a finite Borel measure on $\mathbb{C}$ with compact support $S(\mu) := \text{supp}(\mu)$. Let $\Omega = \Omega(\mu)$ denote the *outer domain* of $S(\mu)$, that is, the unbounded component of $\overline{\mathbb{C}} \setminus S(\mu)$, $\text{Co}(S(\mu))$ the *convex hull*, and $\text{Pc}(S(\mu))$ the *polynomial convex hull* of $S(\mu)$, that is, $\text{Pc}(S(\mu)) = \overline{\mathbb{C}} \setminus \Omega$. (The name polynomial convex hull is derived from the fact that for any compact set $S \subseteq \mathbb{C}$ the set $\text{Pc}(S)$ is the intersection of all sets $S_p := \{z \in \mathbb{C} \mid \ |p(z)| \leq \sup_{x \in S} |p(x)|\}$ with p a polynomial not identically zero.) The set $\partial\Omega \subseteq \partial S(\mu)$ is called the *outer boundary* of $S(\mu)$.

We shall always assume that the support of μ consists of infinitely many points. Then we can form the uniquely existing *orthonormal polynomials*

$$p_n(\mu; z) = \gamma_n(\mu) z^n + \cdots, \qquad \gamma_n(\mu) > 0, \ n \in \mathbb{N},$$

with respect to μ, which are defined by the orthogonality relations

$$\int p_n(\mu; z) \overline{p_m(\mu; z)} d\mu(z) = \delta_{n,m},$$

where $\delta_{n,m} = 1$ if $n = m$ and $\delta_{n,m} = 0$ otherwise. $\gamma_n(\mu)$ is called the *leading coefficient* of $p_n(\mu; \cdot)$.

Whereas $\text{cap}(S(\mu))$ and $g_\Omega(z; \infty)$ depend only on the set $S(\mu)$, or more precisely, on the outer boundary $\partial\Omega$ of $S(\mu)$, we now introduce a type of capacity and Green function that depends on the carriers of the measure μ.

Definition 1.1.1. *Let $\Gamma(\mu)$ be the set of all carriers of the measure μ, that is,*

(1.1) $$\Gamma(\mu) := \{C \subseteq \mathbb{C} \mid \ C \text{ a Borel set and } \mu(\mathbb{C} \setminus C) = 0\},$$

then the minimal-carrier capacity *(of the measure μ) is defined as*

(1.2) $$c_\mu := \inf\{\text{cap}(C) \mid \ C \in \Gamma(\mu), \ C \text{ bounded}\}$$

and the minimal-carrier Green function *(of the measure μ) is defined as*

(1.3) $$g_\mu(z; \infty) := \sup\{g_{\overline{\mathbb{C}} \setminus C}(z; \infty) \mid \ C \in \Gamma(\mu), \ C \text{ bounded}\}, \quad z \in \mathbb{C}.$$

Lemma 1.1.2. *We have*

(1.4) $$c_\mu \leq \text{cap}(S(\mu))$$

and

(1.5) $$g_\mu(z; \infty) \geq g_\Omega(z; \infty)$$

for all $z \in \mathbb{C}$.

Proof. The lemma immediately follows from $S(\mu) \in \Gamma(\mu)$. □

The Examples 1.5.1 and 1.5.2 show that in (1.4) and (1.5) proper inequality as well as equality may hold true.

Before we come to the main theorems of this section, we state a lemma about the location of the zeros of the orthonormal polynomials $p_n(\mu; z)$, $n \in \mathbb{N}$. It will be proved together with related materials in Section 2.1.

Lemma 1.1.3. *All zeros of the orthonormal polynomials* $p_n(\mu; z)$, $n \in \mathbb{N}$, *are contained in the convex hull* $\mathrm{Co}(S(\mu))$, *and for any compact set* $V \subseteq \Omega$ *the number of zeros of* $p_n(\mu; z)$, $n \in \mathbb{N}$, *on* V *is bounded as* $n \to \infty$.

Remarks. (1) If $S(\mu) \subseteq \mathbb{R}$, then $\mathrm{Co}(S(\mu))$ is the smallest interval $I \subseteq \mathbb{R}$ containing $S(\mu)$. Lemma 1.1.3 shows that a well-known result from the theory of orthonormal polynomials associated with measures μ with support $S(\mu) \subseteq \mathbb{R}$, namely, that all zeros of $p_n(\mu; z)$, $n \in \mathbb{N}$, are contained in $\mathrm{Co}(S(\mu))$ (see Section 1.2.2 of [Fr]), carries over to weight measures μ with compact support in $\mathbb{C}$. However, the result that all zeros of $p_n(\mu; z)$, $n \in \mathbb{N}$, are simple, which holds in the case of measures μ with $S(\mu) \subseteq \mathbb{R}$, is no longer true in the general case. (See, for instance, the very simple Example 1.5.1.)

(2) In the case $S(\mu) \subseteq \mathbb{R}$ the orthonormal polynomials $p_n(\mu; z)$, $n \in \mathbb{N}$, can have zeros outside $\mathrm{Pc}(S(\mu))$ only if $S(\mu)$ is disconnected. The arc measure on the unit circle and Example 2.1.2 show that for measures μ with support in $\mathbb{C}$ the situation is different. There the support $S(\mu)$ can be a continuum (but not convex), and the orthonormal polynomials $p_n(\mu; z)$ can have zeros outside $\mathrm{Pc}(S(\mu))$. These zeros may even cluster outside of $\mathrm{Pc}(S(\mu))$ as $n \to \infty$. The phenomenon explains why in the next theorem the lower asymptotic bound requires different formulations for each of the three cases $z \notin \mathrm{Co}(S(\mu))$, $z \in \Omega \cap \mathrm{Co}(S(\mu))$, and $z \in \partial\Omega$.

For the formulation of our main result we introduce the following limit relation. We say that

$$\liminf_{n\to\infty} |f_n(z)| \geq h(z)$$

holds true *locally uniformly* in an open set D if for every $z \in D$ and $z_n \to z$ as $n \to \infty$ we have

$$\liminf_{n\to\infty} |f_n(z_n)| \geq h(z).$$

Thus, the two notions "locally uniformly in D" and "uniformly on compact subsets of D" are different. In general, the latter is stronger than the former one. However, if h is continuous, or merely upper semicontinuous (for a lim sup relation, h lower semicontinuous), then these two notions coincide.

Theorem 1.1.4. *We have*

$$\limsup_{n\to\infty} |p_n(\mu;z)|^{1/n} \le e^{g_\mu(z;\infty)} \tag{1.6}$$

locally uniformly in $\mathbb{C}$, *and*

$$\liminf_{n\to\infty} |p_n(\mu;z)|^{1/n} \ge e^{g_\Omega(z;\infty)} \tag{1.7}$$

locally uniformly in $\mathbb{C}\setminus \mathrm{Co}(S(\mu))$. *In* $\mathrm{Co}(S(\mu))\cap\Omega$ *the asymptotic lower bound* (1.7) *holds true only in capacity, that is, for every compact set* $V\subseteq\Omega$ *and every* $\varepsilon>0$ *we have*

$$\lim_{n\to\infty} \mathrm{cap}(\{z\in V \mid |p_n(\mu;z)|^{1/n} < e^{g_\Omega(z;\infty)}-\varepsilon\}) = 0. \tag{1.8}$$

In $\mathrm{Co}(S(\mu))\cap\Omega$ *the lower bound can also be given in the following form: For every infinite subsequence* $N\subseteq\mathbb{N}$ *we have*

$$\limsup_{n\to\infty,\ n\in N} |p_n(\mu;z)|^{1/n} \ge e^{g_\Omega(z;\infty)} \qquad \text{qu.e. in } \Omega\cap\mathrm{Co}(S(\mu)), \tag{1.9}$$

and on the outer boundary $\partial\Omega$ *of* $S(\mu)$ *we have*

$$\limsup_{n\to\infty,\ n\in N} |p_n(\mu;z)|^{1/n} \ge 1 \qquad \text{qu.e. on } \partial\Omega. \tag{1.10}$$

Remarks. (1) If the two Green functions $g_\Omega(z;\infty)$ and $g_\mu(z;\infty)$ are identical, then we have proper and identical limits in (1.6) and (1.7). (The existence of a proper limit and equality in (1.7) will be called regular asymptotic behavior in Section 3.1.) If the two Green functions are not identical for a weight measure μ, that is, if we have a proper inequality in (1.5) for some value $z\in\mathbb{C}$, then it will be shown in Theorem 1.1.9 that there exists a measure ν having the same set of carriers as μ, that is, $\Gamma(\nu)=\Gamma(\mu)$ and hence $g_\mu(z;\infty)\equiv g_\nu(z;\infty)$, such that we have equality in both (1.6) and (1.7) for $\{|p_n(\nu;z)|^{1/n} \mid n\in\mathbb{N}\}$. Hence, both bounds are sharp.

(2) Whereas the upper asymptotic bound (1.6) holds true locally uniformly on $\mathbb{C}$, the lower bound (1.7) holds true in this strong sense only on compact subsets of $\mathbb{C}\setminus\mathrm{Co}(S(\mu))$. This weakness of the lower asymptotic bound in $\mathrm{Co}(S(\mu))$ is caused by the zeros that the polynomials $p_n(\mu;z)$ may have everywhere in $\mathrm{Co}(S(\mu))$. In Corollary 1.1.5 the zeros in $\mathrm{Co}(S(\mu))\cap\Omega$ will be factored out.

(3) The upper asymptotic bound (1.6) is not specific for orthonormal polynomials. It holds true for any sequence of polynomials normalized in $L^2(\mu)$, as will be shown in Theorem 1.1.8.

(4) On the outer boundary $\partial\Omega$ of $S(\mu)$ an asymptotic estimate in capacity such as the one in (1.8) cannot, in general, be true since there the asymptotic density of the zeros can be positive almost everywhere.

(5) The simple Example 1.5.1 shows that in the interior of the polynomial convex hull $\mathrm{Pc}(S(\mu))$ we cannot, in general, expect 1 as a lower asymptotic bound. The special case $\mathrm{Int}(\mathrm{Pc}(S(\mu))) = \emptyset$, which includes all weight measures μ on the real axis, is formulated separately in Corollary 1.1.6.

Corollary 1.1.5. *Let $U, V \subseteq \Omega$ be two compact sets, U containing V in its interior, and let $x_{n,1}, \ldots, x_{n,m(n)}$ be the zeros of $p_n(\mu; z)$, $n \in \mathbb{N}$, on U. Then we have*

$$(1.11) \qquad \liminf_{n\to\infty} \left| \frac{p_n(\mu; z)}{\prod_{j=1}^{m(n)} (z - x_{nj})} \right|^{1/n} \geq e^{g_\Omega(z;\infty)}$$

uniformly on V.

Corollary 1.1.6. *If the interior of $\mathrm{Pc}(S(\mu))$ is empty, then for any infinite subsequence $N \subseteq \mathbb{N}$ we have*

$$(1.12) \qquad \limsup_{n\to\infty,\ n\in N} |p_n(\mu; z)|^{1/n} \geq 1 \qquad \text{qu.e. on} \quad S(\mu).$$

Note that the interior of $\mathrm{Pc}(S(\mu))$ is empty exactly when $S(\mu)$ has connected complement and empty interior.

If we consider the orthonormal polynomials $p_n(\mu; z)$, $n \in \mathbb{N}$, near infinity, then from the upper and lower asymptotic bounds (1.6) and (1.7) in Theorem 1.1.4 we immediately deduce upper and lower asymptotic bounds for the nth root of the *leading coefficient* $\gamma_n(\mu)$ of the orthonormal polynomials:

Corollary 1.1.7. *We have*

$$(1.13) \qquad \frac{1}{\mathrm{cap}(S(\mu))} \leq \liminf_{n\to\infty} \gamma_n(\mu)^{1/n} \leq \limsup_{n\to\infty} \gamma_n(\mu)^{1/n} \leq \frac{1}{c_\mu},$$

where $\mathrm{cap}(S(\mu)) = 0$ or $c_\mu = 0$ is allowed.

It has already been mentioned in Remark 3 to Theorem 1.1.4 that the upper asymptotic bound (1.6) is not specific for orthonormal polynomials. In the next theorem we state the result for arbitrary sequences of polynomials. The upper bound (1.6) then follows as a corollary.

Theorem 1.1.8. *For any sequence of polynomials P_n not identically zero and of degree at most $n \in \mathbb{N}$, we have*

$$(1.14) \qquad \limsup_{n\to\infty} \left(\frac{|P_n(z)|}{\|P_n\|_{L^2(\mu)}} \right)^{1/n} \leq e^{g_\mu(z;\infty)}$$

locally uniformly in $\mathbb{C}$.

Finally, we show the sharpness of the estimates in Theorem 1.1.4. To do this let us call two measures ν and μ *carrier related* (denoted $\nu \sim \mu$) if ν and μ have the same carriers, that is, $\Gamma(\nu) = \Gamma(\mu)$. Obviously, this is the same as their mutual absolute continuity on each other. Since for carrier-related measures ν and μ the Green functions g_μ and g_ν are identical, and furthermore $S(\nu) = S(\mu)$, it follows that the upper and lower bounds (1.6) and (1.7) are the same for carrier-related measures. Now, from this point of view they are sharp.

Theorem 1.1.9. (a) *There is a* $\nu_1 \sim \mu$ *such that*

$$\lim_{n\to\infty} |p_n(\nu_1; z)|^{1/n} = e^{g_\mu(z;\infty)} \tag{1.15}$$

locally uniformly for $z \notin \mathrm{Co}(S(\mu))$.
(b) *There is a* $\nu_2 \sim \mu$ *such that*

$$\lim_{n\to\infty} |p_n(\nu_2; z)|^{1/n} = e^{g_\Omega(z;\infty)} \tag{1.16}$$

locally uniformly for $z \notin \mathrm{Co}(S(\mu))$.
(c) *There is a* $\nu_3 \sim \mu$ *such that*

$$\liminf_{n\to\infty} |p_n(\nu_3; z)|^{1/n} = e^{g_\Omega(z;\infty)} \tag{1.17}$$

and

$$\limsup_{n\to\infty} |p_n(\nu_3; z)|^{1/n} = e^{g_\mu(z;\infty)} \tag{1.18}$$

for every $z \notin \mathrm{Co}(S(\mu))$.

The unimprovability of the other estimates in Theorem 1.1.4 can also be verified. Without going into details we only remark concerning (1.8):

Corollary 1.1.10. *With* μ *and* ν_2 *as in Theorem 1.1.9, for every compact set* $V \subseteq \Omega$ *we have*

$$\limsup_{n\to\infty} |p_n(\nu_2; z)|^{1/n} \le e^{g_\Omega(z;\infty)} \tag{1.19}$$

uniformly on V.

1.2 Some Potential-theoretic Preliminaries

Minimal carriers will be introduced in this section. Representations for the minimal-carrier Green function $g_\mu(z;\infty)$ and a related principle of domination will be proved. All results will be used throughout the whole chapter. We start with some terminology.

A sequence of measures $\{\nu_n\}$ is said to *converge weakly* (or in the *weak** topology) to a measure ν (in $\overline{\mathbb{C}}$), written $\nu_n \xrightarrow{*} v$, if for every function f continuous in $\overline{\mathbb{C}}$ we have

$$\int f d\nu_n \to \int f d\nu \quad \text{as} \quad n \to \infty.$$

Since the unit ball of positive measures is weak* compact, from every sequence $\{\nu_n\}$ of probability measures we can select an infinite subsequence with index set $N \subseteq \mathbb{N}$ such that $\{\nu_n \mid n \in N\}$ is weak* convergent. This result is often called Helly's selection theorem.

For a (Borel) measure ν with compact support $S(\nu)$ in $\mathbb{C}$ we set

$$p(\nu; z) := \int \log \frac{1}{|z-x|} d\nu(x), \tag{2.1}$$

and call it the (*logarithmic*) *potential* of ν.

For a polynomial P we denote by ν_P the positive measure that has at every zero of P a mass equal to the multiplicity of that zero. This measure is called the *zero distribution* of P. Thus, $\|\nu_P\| = \deg(P)$ and for monic polynomials P we have $\log|P(z)| = -p(\nu_P; z)$.

The Green function $g_B(z; \infty)$ of a Borel set $B \subseteq \overline{\mathbb{C}}$ with bounded complement has already been mentioned in Section 1.1. In order to have a basis for proofs, we state three fundamental properties of $g_B(z; \infty)$, which can be taken as *defining properties*:

(i) $g_B(z; \infty)$ is nonnegative and subharmonic in $\mathbb{C}$, and harmonic in $\mathrm{Int}(B) \setminus \{\infty\}$.
(ii) $g_B(z; \infty) = \log|z| - \log(\mathrm{cap}(\mathbb{C} \setminus B)) + o(1)$ as $|z| \to \infty$, where $o(1)$ tends to 0 as $|z| \to \infty$.
(iii) $g_B(z; \infty) = 0$ for qu.e. z on $\mathbb{C} \setminus B$.

If $\mathrm{cap}(\mathbb{C} \setminus B) = 0$, then $g_B(z; \infty) \equiv \infty$. The function $g_B(z; \infty)$ exists and is uniquely determined by (i) to (iii) for every Borel set $B \subseteq \overline{\mathbb{C}}$ with bounded complement $\overline{\mathbb{C}} \setminus B$; see Appendix V. We note that we have defined $g_B(z; \infty)$ for all $z \in \mathbb{C}$. If the bounded complement $C := \mathbb{C} \setminus B$ is of positive capacity, that is, if

$$\mathrm{cap}(C) > 0, \tag{2.2}$$

then (see Appendixes IV and V) there uniquely exists a probability measure ω_C, called the *equilibrium distribution* of C, with $S(\omega_C) \subseteq \overline{C}$ such that the Green function $g_B(z; \infty)$ has the representation

$$g_B(z; \infty) \equiv -p(\omega_C; z) - \log(\mathrm{cap}(C)). \tag{2.3}$$

If C is a compact set, then $S(\omega_C)$ is contained in the outer boundary $\partial \mathrm{Pc}(C)$ of C; in general, however, we only know that $S(\omega_C) \subseteq \overline{C}$. Since

ω_C is in general not carried by C, it is sometimes useful to know that the set C can be exhausted from within by nested compact sets $C_n \subseteq C$ so the capacity, the associated Green functions, and the equilibrium distributions converge to the corresponding objects of the set C. More formally:

There exist compact sets $C_n \subseteq C$, $n \in \mathbb{N}$, *with* $C_n \subseteq C_{n+1}$ *and* $\text{cap}(C_n) > 0$ *such that*

$$\lim_{n\to\infty} \text{cap}(C_n) = \text{cap}(C), \tag{2.4}$$

$$\lim_{n\to\infty} g_{\overline{\mathbb{C}}\setminus C_n}(z;\infty) = g_{\overline{\mathbb{C}}\setminus C}(z;\infty) \tag{2.5}$$

for all $z \in \mathbb{C}$, *and* $\omega_{C_n} \xrightarrow{*} \omega_C$ *as* $n \to \infty$. *The sequence* $\{g_{\overline{\mathbb{C}}\setminus C_n}(z;\infty)\}$ *is monotonically decreasing.*

We will only sketch the proof: From the capacitability of Borel sets (see Appendix I) it follows that a sequence $\{C_n\}$ exists so that (2.4) holds true. The existence of limit (2.5) and the weak* limit of the equilibrium distributions follows from the monotonicity of the sequence $\{g_{\overline{\mathbb{C}}\setminus C_n}(z;\infty)\}$ (see Theorem 1.26 of [La]). That the limit (2.5) is identical with the Green function $g_B(z;\infty)$ can then be derived with the help of the three defining properties of Green functions by standard techniques (see also the reasonings applied in Appendix IV and V).

We note that all sets and functions to be subsequently defined are Borel measurable and therefore we do not have to care about inner and outer capacity.

After these general results from potential theory we turn to results related to the minimal-carrier Green function $g_\mu(z;\infty)$.

The elements of $\Gamma(\mu)$ are partially ordered by inclusion and the capacity is a monotone set function. It is therefore possible and often useful to consider elements in $\Gamma(\mu)$ which are minimal with respect to capacity.

Definition 1.2.1. *The elements of the set*

$$\Gamma_0(\mu) := \{C \in \Gamma(\mu) \mid \text{cap}(C) = c_\mu \quad \textit{and} \quad C \subseteq S(\mu)\} \tag{2.6}$$

are called minimal carriers.

Lemma 1.2.2. *The set* $\Gamma_0(\mu)$ *is not empty, and for every* $C \in \Gamma(\mu)$ *there exists an element* $C_0 \in \Gamma_0(\mu)$ *with* $C_0 \subseteq C$.

Remark. Contrary to the minimal-carrier capacity c_μ and the minimal-carrier Green function $g_\mu(z;\infty)$, which are both uniquely defined, there exist in general more than one minimal element in $\Gamma_0(\mu)$.

Proof. Let $C \in \Gamma(\mu)$ be arbitrary. By (1.2) there exists a sequence

$C_n \in \Gamma(\mu)$, $n \in \mathbb{N}$, with $\operatorname{cap}(C_n) \to c_\mu$ as $n \to \infty$. If we set $C_{0n} := S(\mu) \cap C_n$, $n \in \mathbb{N}$, then we have $C_{0n} \in \Gamma(\mu)$ for all $n \in \mathbb{N}$, and therefore

$$(2.7) \qquad C_0 := \bigcap_{n=1}^{\infty} C_{0n} \in \Gamma(\mu) \quad \text{and} \quad \operatorname{cap}(C_0) = c_\mu.$$

Hence, $C_0 \in \Gamma_0(\mu)$ and $C_0 \subseteq C$. □

The minimal carriers of $\Gamma_0(\mu)$ provide us with a representation of the minimal-carrier Green function $g_\mu(z;\infty)$ as an ordinary Green function.

Lemma 1.2.3. *For any $C \in \Gamma_0(\mu)$ we have*

$$(2.8) \qquad g_{\overline{\mathbb{C}}\setminus C}(z;\infty) \equiv g_\mu(z;\infty),$$

and

$$(2.9) \qquad g_\mu(z;\infty) = 0 \qquad qu.e. \text{ on } C.$$

Proof. If $c_\mu = 0$, then $g_{\overline{\mathbb{C}}\setminus C}(z;\infty) \equiv \infty$ and $\operatorname{cap}(C) = 0$ for all $C \in \Gamma_0(\mu)$. Since in this case we also have $g_\mu(z;\infty) \equiv \infty$, the identities (2.8) and (2.9) hold true.

Let us now assume $c_\mu > 0$, and let C_1 and C_2 be two arbitrary elements of $\Gamma_0(\mu)$. $C_0 := C_1 \cap C_2$ also belongs to $\Gamma_0(\mu)$, and therefore $\operatorname{cap}(C_j) = c_\mu$ for $j = 0, 1, 2$. By checking the three defining properties (i) to (iii) of a Green function, it is easy to verify that both Green functions $g_{\overline{\mathbb{C}}\setminus C_j}(z;\infty)$, $j = 1, 2$, are at the same time the Green function of the set $\overline{\mathbb{C}} \setminus C_0$. Hence, by the uniqueness of Green functions we have

$$(2.10) \qquad g_{\overline{\mathbb{C}}\setminus C_1}(z;\infty) \equiv g_{\overline{\mathbb{C}}\setminus C_2}(z;\infty),$$

which shows that all Green functions $g_{\overline{\mathbb{C}}\setminus C}(z;\infty)$, $C \in \Gamma_0(\mu)$, are identical.

Since for two bounded Borel sets $C_1, C_2 \subseteq \mathbb{C}$ the inclusion $C_1 \subseteq C_2$ implies

$$g_{\overline{\mathbb{C}}\setminus C_1}(z;\infty) \geq g_{\overline{\mathbb{C}}\setminus C_2}(z;\infty) \quad \text{for all} \quad z \in \mathbb{C},$$

and since every carrier $C \in \Gamma(\mu)$ contains a minimal carrier $C_0 \in \Gamma_0(\mu)$, identity (2.9) follows from identity (2.10) and definition (1.3) of the minimal-carrier Green function $g_\mu(z;\infty)$. Equality (2.9) then follows from the corresponding property of $g_{\overline{\mathbb{C}}\setminus C}(z;\infty)$. □

The next lemma immediately follows from Lemma 1.2.3 together with representation (2.3) for ordinary Green functions.

Lemma 1.2.4. *If $c_\mu > 0$, then all $C \in \Gamma_0(\mu)$ possess the same equilibrium distribution, which we denote by ω_μ, and for $g_\mu(z;\infty)$ we have the representation*

$$(2.11) \qquad g_\mu(z;\infty) \equiv -p(\omega_\mu; z) - \log c_\mu.$$

Definition 1.2.5. *The probability measure* ω_μ *is called the* minimal-carrier equilibrium distribution *associated with* μ.

Lemma 1.2.6. *We have*

$$g_\Omega(z;\infty) \equiv g_\mu(z;\infty) \tag{2.12}$$

if and only if

$$c_\mu = \mathrm{cap}(S(\mu)). \tag{2.13}$$

Proof. Let us assume (2.12). Then (2.13) follows from considering both Green functions in (2.12) near infinity.

Let us now assume (2.13). Then $S(\mu)$ is a minimal carrier and (2.12) is a consequence of (2.8). □

Next we investigate the special case when the weight measure μ is equal to the equilibrium distribution $\omega_{S(\mu)}$ of $S(\mu)$. We shall need the results later. But besides that, the lemma shows that the equilibrium distribution is an instructive example of a weight measure μ that satisfies (2.13).

Lemma 1.2.7. *Let* $S \subseteq \mathbb{C}$ *be a compact set with* $\mathrm{cap}(S) > 0$, *and set* $\omega := \omega_S$. *Then we have*

$$c_\omega = \mathrm{cap}(S) = \mathrm{cap}(S(\omega)), \tag{2.14}$$

$$g_\omega(z;\infty) \equiv g_{\overline{\mathbb{C}}\setminus S}(z;\infty), \tag{2.15}$$

and

$$\omega_\omega = \omega. \tag{2.16}$$

Proof. If we represent $g_\Omega(z;\infty)$, $\Omega := \mathbb{C} \setminus \mathrm{Pc}(S(\omega))$, and $g_\omega(z;\infty)$ as in (2.3) and (2.11), respectively, by logarithmic potentials and apply Fubini's theorem, then we get

$$\int g_\omega(z;\infty)d\omega(z) = \int g_\Omega(z;\infty)d\omega_\omega(z) + \log \frac{\mathrm{cap}(S(\omega))}{c_\omega}. \tag{2.17}$$

(For the case $\mathrm{cap}(S(\omega)) = c_\omega = 1$, equality (2.17) is known in potential theory as the formula of reciprocity.)

Let C_0 be the set of all $z \in \mathbb{C}$ with $g_\omega(z;\infty) = 0$. Then from Lemma 1.2.3 we know that $\mathrm{cap}(C \setminus C_0) = 0$ for any $C \in \Gamma_0(\omega)$. Since $\omega = \omega_S$ is of finite energy it follows that $\omega(C \setminus C_0) = 0$ and therefore $C_0 \in \Gamma(\omega)$. Hence, the integral on the left-hand side of (2.17) is zero. The integral on the right-hand side of (2.17) is nonnegative, which implies $c_\omega \geq \mathrm{cap}(S(\omega))$, and with inequality (1.4), formula (2.14) follows.