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1

Asymptotic Behaviour of Anisotropic Wave Systems Stimulated by Oscillating Sources

Sir James Lighthill
University College London

1.1 Abstract

The theory of the far-field asymptotic behaviour of anisotropic waves, forced by a local distribution of sources of fixed frequency ω_0, expresses those waves at a given distant location in the form of an integral over just a part of the wavenumber surface S (the surface in wavenumber space that is specified, for frequency ω_0, by the dispersion relationship). In the theory as set out by Lighthill (1960, 1978) the integral is taken over S_+, defined as that part of S for which the dispersion relationship makes the gradient of frequency in wave-number space have a positive component (this is a form of "the radiation condition") in the direction of the given distant location.

On the other hand a recent more rigorous treatment (Lighthill 1990) has shown that the integral is more correctly taken over a combination of the open surface S_+ (defined as the above part of the real wavenumber surface S) and S_{-i} (defined as a part of the complex wavenumber surface obtained by proceeding from the edge C of S_+ in a negative pure-imaginary direction). The consequent suppression of any discontinuity at C is important, as eliminating all possibility of an associated spurious $O(x^{-1})$ asymptotic signal at large distances x from the source region.

The new rigorous analysis confirms the correctness of the results obtained (Lighthill 1978) by ignoring any possible contribution from C; in other words, by simply applying stationary-phase principles to the integral over S_+. At the end of the paper, some of these results are recalled briefly, together with their application to a particular problem of wave asymptotics in which a powerful interest has been taken by Ursell (1960) and also by Inui (1962): the analysis of ship waves and of approaches towards reducing the wavemaking resistance of ships.

1.2 Introduction

At this Retirement Meeting honouring a great exponent of fluid dynamics, of waves and of asymptotics, who since 1961 held Manchester University's Beyer Chair of Applied Mathematics, I happily seize an early

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opportunity to make a remark both on my own behalf and simultaneously, perhaps, for all those other former holders of the Beyer Chair (from Horace Lamb to Sydney Goldstein) who are with us no longer. In this brief preliminary remark I want to voice a view which (in some mystic sense!) I confidently claim as our collective view of how magnificently the traditions of this famous Chair have during the past three decades been maintained – in fluid dynamics, in waves, in asymptotics – by the eminent tenure of Fritz Ursell.

And next, like most of the speakers at this meeting, I proceed to offer a paper on a favourite theme (see title above) within the broad field that may be described as *Wave Asymptotics:* a paper in which, while referring to earlier treatments of my theme by Lighthill (1960, 1978), I comprehensively reconsider this theme's fundamentals. Indeed, in a certain sense I pay homage rather specifically to Fritz Ursell by reconsidering my theme with a highly necessary injection of *added rigour* !

I shall present, then, a reconsideration of the asymptotics of forced vibrations in anisotropic wave systems that are *homogeneous*; being described, indeed, by linear partial differential equations with constant coefficients. The systems are subjected to local forcing (specified by a right-hand side of the partial differential equation that is nonzero over just a limited region of space) at fixed radian frequency ω_0.

In relation to my choice of topic I may perhaps make four general points, as follows.

(a) Anisotropic wave propagation is widespread and important.

(b) It is easy (see §1.11 below) to extend the theory to the case of travelling forcing effects; where "Doppler shifts" make a problem effectively anisotropic even in cases where the wave system as such is isotropic.

(c) The theory is readily adapted (see §1.3 below) to cases of boundary forcing.

(d) The theory for homogeneous systems shows wave energy travelling away from the local forcing region along straight rays, but it can be extended to systems with gradually varying properties (leading to gradually varying coefficients of the partial differential equations) by the refraction theory for rays (see Lighthill 1978, section 4.5); which states that, in general linear systems with a dispersion relationship

$$\omega = \omega(k_i, x_i) \tag{1.1}$$

that links frequency ω to vector wavenumber k_i at the point with position vector x_i, the rays are described by equations in Hamiltonian form,

$$\frac{dx_i}{dt} = \frac{\partial \omega}{\partial k_i}, \frac{dk_i}{dt} = -\frac{\partial \omega}{\partial x_i}, \tag{1.2}$$

implying that wave energy travels at the group velocity $\partial\omega/\partial k_i$ appropriate to its wavenumber k_i, which itself varies gradually by refraction.

In the present lecture, however, I limit myself to homogeneous systems for which the dispersion relationship takes a simpler form,

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$$\omega = \omega(k_i), \tag{1.3}$$

that allows wave energy to travel along straight rays with unchanging wavenumber.

1.3 Explanation of the Fourier integral which has to be asymptotically estimated

I consider local forcing by a source distribution $e^{i\omega_0 t} f(\mathbf{x})$, where the forcing amplitude $f(\mathbf{x})$ has Fourier transform $F(\mathbf{k})$ so that it can be written as a Fourier integral

$$f(\mathbf{x}) = \int F(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k}, \tag{1.4}$$

taken over the space of all wavenumbers

$$\mathbf{k} = (k_1, k_2, k_3). \tag{1.5}$$

Then the mathematical problem which has to be solved is to estimate another Fourier integral,

$$q = e^{i\omega_0 t} \int \frac{F(\mathbf{k})}{B(\omega_0, \mathbf{k})} e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k}, \tag{1.6}$$

in which the denominator B represents a function of ω and $\mathbf{k}$ such that the equation

$$B(\omega, \mathbf{k}) = 0 \tag{1.7}$$

is a form of the dispersion relationship (1.3).

The integral expression (1.6) is derived in three dimensions (the case here treated) from a partial differential equation

$$\begin{aligned} B\left(-i\frac{\partial}{\partial t}, i\frac{\partial}{\partial x_1}, i\frac{\partial}{\partial x_2}, i\frac{\partial}{\partial x_3}\right) q &= e^{i\omega_0 t} f(\mathbf{x}) \\ &= e^{i\omega_0 t} \int F(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k}, \end{aligned} \tag{1.8}$$

where B is a polynomial – or, more generally, a rational function – in its four variables. Essentially this is because the Fourier transform $e^{i\omega_0 t} Q(\mathbf{k})$ of q must satisfy

$$B(\omega_0, \mathbf{k}) Q = F, \tag{1.9}$$

from which equation (1.6) follows immediately.

In two dimensions an identical integral expression can, needless to say, be obtained from an equation similar to (1.9) but with only the three independent variables t, x_1 and x_2. On the other hand, there is also a second, perhaps more interesting, way in which equation (1.6) may arise in two dimensions.

This is the case of boundary forcing at a plane boundary $x_3 =$ constant, where a certain quantity η is subjected to local forcing

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$$\eta = e^{i\omega_0 t} f(x_1, x_2) = e^{i\omega_0 t} \int F(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k} \tag{1.10}$$

at frequency ω_0, the integration being over the space of all wavenumbers $\mathbf{k} = (k_1, k_2)$. Here, the three-dimensional system is supposed to satisfy a linear partial differential equation with constant coefficients and zero right-hand side; furthermore, the condition $\eta = 0$ (corresponding to the case when the boundary is free) yields the dispersion relationship (1.7). Finally, the quantity η (representing local forcing) is related to a basic independent variable $q(t, x_1, x_2)$ in such a way that

$$B(\omega_0, k_1, k_2) Q = F \tag{1.11}$$

for the Fourier transform $e^{i\omega_0 t} Q(\mathbf{k})$ of q; a fact which directly yields the integral expression (1.6) for q.

At a meeting in Fritz Ursell's honour it is apposite to use *water waves* (say, gravity waves on the surface of deep water) to illustrate boundary forcing. If q is surface elevation then η, the surface pressure (relative to atmospheric), is given on linear theory as

$$\eta = -\rho g q - \rho \partial \phi / \partial t \tag{1.12}$$

where ρ is fluid density and where the velocity potential ϕ satisfies a boundary condition

$$\partial q / \partial t = \partial \phi / \partial x_3 \tag{1.13}$$

at the free surface's undisturbed position. On the other hand, Laplace's equation for motions in deep water (which vanish as $x_3 \to -\infty$) implies that waves of horizontal wave number (k_1, k_2) must everywhere satisfy

$$\partial \phi / \partial x_3 = -\left(k_1^2 + k_2^2\right)^{1/2} \phi; \tag{1.14}$$

which, with equations (1.12) and (1.13), implies that the Fourier transform of q satisfies equation (1.11) with

$$B(\omega, k_1, k_2) = \rho \omega^2 (k_1^2 + k_2^2)^{-1/2} - \rho g. \tag{1.15}$$

Then, as expected, the equation (1.7) representing the case of a free boundary is a form of the classical dispersion relationship

$$\omega = (gk)^{1/2} = g^{1/2} (k_1^2 + k_2^2)^{1/4}; \tag{1.16}$$

on the other hand, the forced boundary condition (1.10) describes the application of oscillatory pressure sources at the surface which generate outgoing waves given by the integral expression (1.6).

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1.4 Special problems in the estimation of this Fourier integral

Whether in two or three dimensions there is an important and famous difficulty in estimating the Fourier integral (1.6). This is that the integral is mathematically ambiguous whenever the denominator $B(\omega_0, \mathbf{k})$ possesses zeros for real $\mathbf{k}$: zeros which, in general, are poles of the integrand[1].

Thus, whenever the integral with respect to k_1 (say) encounters such a pole for real k_1, we are constrained to ask whether we should take the integral to the right or to the left of the pole; or, alternatively, to use neither version exclusively but a linear combination of both. (The most famous such linear combination – which, I must however emphasize, does not prove to be the correct answer in the present problem – is the Cauchy Principal Value; namely, the arithmetic mean of the two versions.)

This mathematical ambiguity has, as is well known, a physical counterpart. A wave system, compatible with the given forcing which acts within a local region, might in principle be found to incorporate also free waves with their energy coming in "from infinity". (In a linear system, the linear combination of such free waves with the waves that are truly forced by the local sources would still satisfy the same partial differential equation.)

On physical grounds, however, we need to use that unique mathematical determination of the Fourier integral (1.6) which completely excludes any such free waves with energy coming in from infinity. Lighthill (1960) introduced a physically very natural approach to this determination of the waves truly and exclusively forced by the local sources when he proposed that the problem be replaced by an initial-value problem; namely, that of finding the waves in an initially undisturbed system excited by a local forcing that has gradually grown up from zero to its present value.

Specifically, he proposed a substitution of the local forcing function

$$e^{i\omega_0 t} f(\mathbf{x}) \quad \text{by} \quad e^{\epsilon t} e^{i\omega_0 t} f(\mathbf{x}) = e^{i(\omega_0 - i\epsilon)t} f(\mathbf{x}) \tag{1.17}$$

where ϵ is a small positive number. The substitution (1.17) allows the sources to have grown exponentially (from zero when $t = -\infty$) at a slow rate ϵ which may subsequently be allowed to tend to zero.

It might of course have been thought more natural to give the sources a "sudden switch-on" at time $t = 0$ and determine the subsequent behaviour for large t; the substitution (1.17) for the source function would then become a change to $H(t)e^{i\omega_0 t} f(\mathbf{x})$ where $H(t)$ is the Heaviside unit function. However, because the spectrum of $H(t)e^{i\omega_0 t}$ involves essentially all frequencies – including frequencies at which the system may possibly respond to forcing unstably, or in some other way different from its

[1] Essentially, this happens whenever the dispersion relationship (1.7) has solutions for $\omega = \omega_0$ representing unattenuated waves

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response when sources at frequency ω_0 excite it – the approach to our estimation problem via a "sudden switch-on" involves grave inconveniences; some of which are familiar already in the relatively simple case of gravity waves on deep water from the intricate complexities of the Cauchy-Poisson problem.

We should also reject any approach to the estimation of the Fourier integral (1.6) which simply assumes that the ray theory (1.2) must asymptotically be satisfied; as is taken for granted in certain ways of applying "a radiation condition". Our aim, rather, should be to demonstrate the existence of an asymptotic behaviour consistent with ray theory from a rigorous estimate of the integral.

After rejecting those two alternative procedures, then, we return to the substitution (1.17) which, essentially, replaces ω_0 by $\omega_0 - i\epsilon$ so that the integral expression (1.6) which has to be asymptotically estimated becomes

$$q = e^{i(\omega_0 - i\epsilon)t} \int \frac{F(\mathbf{k})}{B(\omega_0 - i\epsilon, \mathbf{k})} e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k}. \tag{1.18}$$

The modified integral expression (1.18) is in general unambiguously defined for small positive ϵ while its limiting value as ϵ becomes vanishingly small represents the waves actually generated in the local forcing region.

1.5 The estimation method of Lighthill (1960, 1978) briefly recapitulated

The method introduced by Lighthill (1960) for estimating the integral (1.18) utilises an early switch to new Cartesian axes, chosen so as to facilitate the calculation. Then, when estimation is completed, the results are thrown into a form invariant under such change of axes, so that they are valid equally in the original system of axes.

Specifically, with an origin in the midst of the source region, the method studies an arbitrary line ℓ stretching radially out from that origin and seeks to estimate the integral on ℓ at large distances from the origin. To this end, new axes are chosen with ℓ as the positive x_1-axis. At a point on ℓ, then, we have $x_2 = x_3 = 0$ so that $\mathbf{k} \cdot \mathbf{x} = k_1 x_1$; accordingly, the integral expression (1.18) becomes

$$q = e^{i(\omega_0 - i\epsilon)t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_2 dk_3 \int_{-\infty}^{\infty} \frac{F(\mathbf{k}) e^{-ik_1 x_1} dk_1}{B(\omega_0 - i\epsilon, \mathbf{k})} \tag{1.19}$$

which must be estimated as $x_1 \to +\infty$. (In a two-dimensional problem, the k_3 integration in (1.19) would be suppressed.)

In equation (1.19) the inner integral

$$\int_{-\infty}^{\infty} \frac{F(\mathbf{k}) e^{-ik_1 x_1} d\mathbf{k}_1}{B(\omega_0 - i\epsilon, \mathbf{k})} \tag{1.20}$$

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is estimated by lowering the path of integration in the complex k_1-plane (Figure 1.1) so as to give the imaginary part of k_1 a negative value

$$\operatorname{Im} k_1 = -\kappa_1. \tag{1.21}$$

Even if κ_1 be of modest magnitude, the integral (1.20) along the new path (1.21) includes the factor $e^{-\kappa_1 x_1}$ which becomes very small as $x_1 \to +\infty$; when, indeed, the modulus of (1.20) must be less than or equal to the integrand of the integrand's modulus, namely

$$e^{-\kappa_1 x_1} \int_{-i\kappa_1-\infty}^{-i\kappa_1+\infty} \frac{|F(\mathbf{k})|d\kappa_1}{|B(\omega_0 - i\epsilon, \mathbf{k})|} = O(e^{-\kappa_1 x_1}). \tag{1.22}$$

Now Cauchy's theorem tells us that the difference between the integral (1.20) and the same integral taken along this different path (1.21) – so as to give it such an $O(e^{-\kappa_1 x_1})$ value – is equal to

$$(-2\pi i) \times \text{sum of residues at poles with } -\kappa_1 < \operatorname{Im} k_1 < 0. \tag{1.23}$$

Thus, with an $O(e^{-\kappa_1 x_1})$ error, the inner integral (1.20) is given by the expression (1.23); where the first factor is $(-2\pi i)$ because it is in the negative sense that the difference integral encircles those poles.

In his calculation of expression (1.23), Lighthill (1960, 1978) confined his consideration to poles which for $\epsilon = 0$ are on the real k_1-axis but are

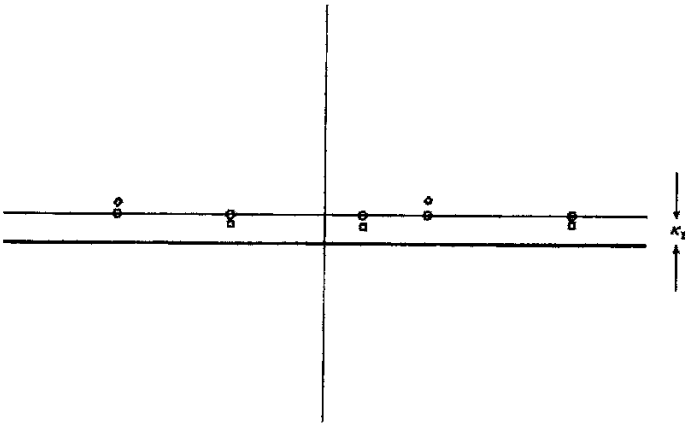

Figure 1.1: A lowered path of integration (thick line) in the complex k_1-plane for the inner integral (1.20). ○ Positions of poles when $\epsilon = 0$; □ Displaced position, when ω_0 is replaced by $\omega_0 - i\epsilon$ with $\epsilon > 0$, of poles k_1 such that $\partial\omega/\partial k_1 > 0$; ◇ Displaced position of poles k_1 such that $\partial\omega/\partial k_1 < 0$. Only the poles □ contribute a residue term when the path is lowered.

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shifted off it for $\epsilon = 0$. Although a reconsideration of this issue will be given in §1.6, we here outline the analysis based on that original approach.

Lighthill (1960, 1978) pointed out that a solution of $B(\omega_0, \mathbf{k}) = 0$ for k_1 real becomes (see Figure 1.1) a solution of $B(\omega_0 - i\epsilon, \mathbf{k}) = 0$ with $\mathrm{Im} k_1 < 0$ if and only if the dispersion relationship $B(\omega, \mathbf{k}) = 0$ for $\mathbf{k}$ real defines a function $\omega = \omega(\mathbf{k})$ such that

$$\partial\omega/\partial k_1 > 0. \tag{1.24}$$

To this mathematically evident statement, the physical counterpart is that the only waves with frequency ω_0 that are expected to appear are those whose energy flow velocity has a positive component $\partial\omega/\partial k_1$ along the x_1-axis (implying that they could have been generated in the source region rather than coming in "from infinity").

In the limit as $\epsilon \to 0$, then, the expression (1.23) becomes $(-2\pi i)$ times the sum of residues of the inner integrand (1.20) at those poles which for $\epsilon = 0$ are on the real k_1-axis and where also the inequality (1.24) is satisfied. From any such pole, that is from a value of k_1 such that

$$\omega(\mathbf{k}) = \omega_0 \quad \text{and} \quad \partial\omega/\partial k_1 > 0, \tag{1.25}$$

the contribution to the inner integrand is

$$\frac{(-2\pi i)F(\mathbf{k})}{\partial B(\omega_0, \mathbf{k})/\partial k_1} e^{-ik_1 x_1} \tag{1.26}$$

by the usual rule for calculating a residue.

Finally, then, with an error $O(e^{-\kappa_1 x_1})$, the complete integral (1.19) is estimated in the limit $\epsilon \to 0$ as

$$q = e^{i\omega_0 t} \int_{S_+} \frac{(-2\pi i)F(\mathbf{k})}{\partial B(\omega_0, \mathbf{k})/\partial k_1} e^{-ik_1 x_1} dk_2 dk_3. \tag{1.27}$$

Here, the outer integration with respect to k_2 and k_3 is confined to a surface S_+ defined as the set of all real wavenumber vectors $\mathbf{k}$ where the two conditions (1.25) are satisfied. Explaining this notation, we may remark that S is used for complete wavenumber surface $\omega(\mathbf{k}) = \omega_0$ while the part of S on which $\partial\omega/\partial k_1$ is positive is called S_+. The method of Lighthill (1960, 1978) concludes by estimating the integral (1.27) over S_+ by classical stationary-phase principles (involving contributions from just those points on S_+ where k_1 is stationary as a function of k_2 and k_3) as will be briefly recalled in §1.10 below – and then applied to a problem (§1.11) which was strongly interesting Fritz Ursell just before his appointment to the Beyer Chair (Ursell 1960).

1.6 Reconsideration of what poles lie in the strip

In the meantime I reconsider the issue of whether the enumeration (1.23) of poles in a narrow strip beneath the real k_1-axis includes any

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poles other than those just described; namely, poles which are on the real k_1-axis for $\epsilon = 0$ but which satisfy the inequality (1.24) so that they lie in the strip for $\epsilon > 0$. Clearly, other poles (if they exist) must be values of k_1 which even for $\epsilon = 0$ have

$$-\kappa_1 < \operatorname{Im} k_1 < 0. \tag{1.28}$$

Simultaneously they must satisfy

$$\omega(\mathbf{k}) = \omega_0; \text{ that is, } B(\omega_0, \mathbf{k}) = 0 \tag{1.29}$$

for ω_0 real (and, of course, for real k_2 and k_3); while the inequality (1.28) shows them to lie, not on the real wavenumber surface, but (for modest-sized κ_1) on a closely neighbouring part of the whole complex wavenumber surface S which (1.29) defines.

Using this remark as a clue, we are guided to look for the remaining poles on such straight lines

$$k_2 = \text{constant}, \; k_3 = \text{constant} \; (k_2, k_3 \text{ real}) \tag{1.30}$$

as may be described as "just missing" the real wavenumber surface S. Figure 1.2 illustrates two such lines; as the equation (1.29) for S shows, they must be lines on which $B(\omega_0, k_1, k_2, k_3)$ as a function of k_1 has *either* a minimum that is positive but very small or a maximum that is a very small negative number. In either case, evidently, a modest pure-imaginary change in k_1 from its value at the stationary point can give $B = 0$. In general this procedure will yield two poles (complex conjugates) depending on whether the change in $\operatorname{Im} k_1$ is positive or negative, but only the pole with $\operatorname{Im} k_1$ negative satisfies the condition (1.28), thus making an additional asymptotic contribution to the inner integral: a contribution first analysed by Lighthill (1990) in a detailed study of which a summary version is now given.

Figure 1.2 suggests that this contribution is associated with the edge C of the open surface S_+ defined by (1.25). This curve C which forms the edge of S_+ must be a locus on which

$$\partial\omega/\partial k_1 = 0 \tag{1.31}$$

so that the normal to S is perpendicular to the k_1-axis. Near a point P on C there can be lines (1.30) which "just miss" intersecting the real wavenumber surface S, yet for which there may in the strip (1.28) be a pole

$$k_1 = k_1^{\mathrm{P}} - im, \quad \text{with} \quad 0 < m < \kappa_1. \tag{1.32}$$

We may describe such a pole (1.32) as lying on a certain complex part S_{-i} of the wavenumber surface; specifically, a part extending from the edge C of S_+ in the negative pure-imaginary k_1-direction.

We proceed now to compare the asymptotic contributions from S_+ and from S_{-i} to our integrals. Intriguingly, we find these contributions to display an elegant mutual complementarity.

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1.7 Formulating the contribution from S_{-i} to our integrals

We first review the local geometry of S (see Figure 1.2) near a point P on that curve C which is defined by equation (1.31) making the normal to S perpendicular to the k_1-axis. Near P we display in Figure 1.2 the intersection of S by a plane parallel to the k_1-direction and including the vector $\mathbf{n}$, which we specify as the normal to S *towards which* the said intersection is convex.

In this plane, if we consider a small vector displacement from P whose component along $\mathbf{n}$ is h, we are able to approximate B at points on $\mathbf{n}$ by

$$(\partial B/\partial n)^P h \tag{1.33}$$

and at general points by

$$(\partial B/\partial n)^P[h + \frac{1}{2}R^{-1}(k_1 - k_1^P)^2], \tag{1.34}$$

where R is the radius of curvature at P of the plane section of S shown in Figure 1.2. This approximate expression (1.34) for B tells us firstly that a pole $B = 0$ takes the form (1.32) with, approximately,

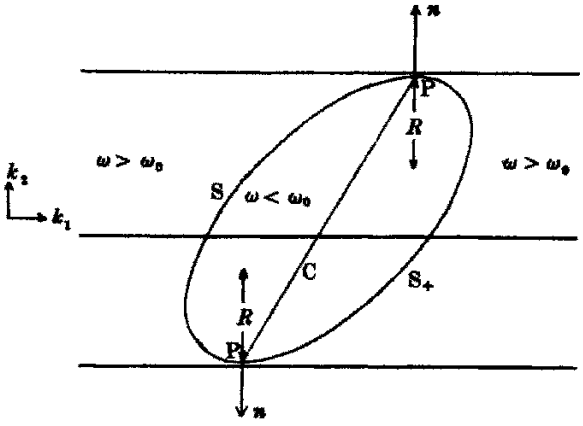

Figure 1.2: A particular wavenumber surface S on which $\omega = \omega_0$; a case when $\omega < \omega_0$ inside S and $\omega > \omega_0$ outside S is shown. That part S_+ of S on which $\partial\omega/\partial k_1 > 0$ is bounded by the curve C on which $\partial\omega/\partial k_1 = 0$. Three horizontal lines, defined as in the equations (1.30), are shown. The middle line intersects S at two real points, out of which only one is an admissible pole because it is on S_+. Each of the other two lines just misses intersecting S at any real point; but it intersects S at a pair of complex conjugate points out of which the point (1.32) is an admissible pole. Here, at the point P on C, the vector $\mathbf{n}$ is defined as the normal to S towards which its intersection by a plane including $\mathbf{n}$ and the k_1-direction is convex (with radius of curvature R).

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