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Mohan S. Shrikhande and Sharad S. Sané

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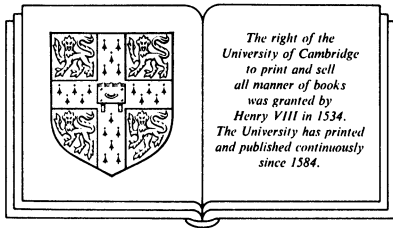
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To Neelima and Aditi, from Mohan

To B.V. Limaye, from Sharad

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PREFACE

Combinatorics is generally concerned with counting arrangements within a finite set. One of the basic problems is to determine the number of possible configurations of a given kind. Even when the rules specifying the configuration are relatively simple, the questions of existence and enumeration often present great difficulties. Besides counting, combinatorics is also concerned with questions involving symmetries, regularity properties, and morphisms of these arrangements. The theory of block designs is an important area where these facts are very apparent. The study of block designs combines number theory, abstract algebra, geometry, and many other mathematical tools including intuition. In the words of G.C. Rota (in: *Studies in Combinatorics*, Mathematical Association of America, 1978),

"Block designs are generally acknowledged to be the most complex mathematical structures that can be defined from scratch in a few lines. Progress in understanding and classification has been slow and proceeded by leaps and bounds, one ray of sunlight followed by years of darkness. ...This field has been enriched and made even more mysterious, a battleground of number theory, projective geometry and plain cleverness. This is probably the most difficult combinatorics going on today..."

In the last few years, some new text-books (Beth, Jungnickel and Lenz, Hughes and Piper, Wallis) on Design Theory have been published. Dembowski's 'Finite Geometries', M. Hall Jr.'s 'Combinatorial Theory,' and Ryser's 'Combinatorial Mathematics' are regarded as some of the classic references in combinatorics, particularly in the area of designs. Block designs have connections with group

theory, graph theory, coding theory, and number theory. The monographs of Biggs and White; Cameron and van Lint are mostly devoted to the developments in specialized topics and show how progress in one of these areas impacts design theory. In spite of the appearance of some recent monographs (particularly the ones by Lander on "Symmetric Designs: An Algebraic Approach", Payne and Thas on "Generalized Quadrangles", Batten on "Combinatorics of Finite Geometries") we believe that not many monographs dealing with special topics in design theory are available at present. This is one of the motivations for writing the present monograph.

As is well known, non-trivial designs satisfy the inequality $v \leq b$, i.e., the number of blocks is at least as large as the number of points. The situation of equality is called a symmetric design. This is an important class of designs which is characterized by the property that the design has precisely one (block) intersection number. Quasi-symmetric designs are designs (i.e., 2-designs) with at most two intersection numbers. With this general definition, symmetric designs are just improper quasi-symmetric designs (q.s. designs). The theory of symmetric designs is mathematically enriched by results such as the Bruck-Ryser-Chowla theorem and Lander's monograph is devoted to this aspect of symmetric designs. No similar strong tools seem to be known for proper quasi-symmetric designs. There have, however, been recent attempts in this direction, mainly by Calderbank, using tools from coding theory. On the other hand, much investigation on (proper) q.s. designs is facilitated by the block graph associated with such a design: If x and y are the two intersection numbers, then make

two vertices (blocks) adjacent if the corresponding blocks intersect in x points. This graph turns out to be a strongly regular graph and is non-trivial if the design is not symmetric. This early observation paved the way for an intersection between theories of quasi-symmetric designs and strongly regular graphs. The latter objects were defined by Bose and the most elegant examples are provided by groups with rank three permutation action. Though many expository articles on strongly regular graphs exist, no comprehensive book dealing with all the aspects of this topic seems to be available at present. This fact has also been mentioned in the very recent book on distance regular graphs by Brouwer, Cohen, and Neumaier.

At this stage, we discuss the organization of the chapters in this monograph. The material is divided into ten chapters, the first three of which cover basic material on designs, strongly regular graphs, and quasi-symmetric designs respectively. Various standard examples of symmetric designs (Hadamard designs and projective geometries in particular), strongly regular graphs, and quasi-symmetric designs (affine designs in particular) are included. Our treatment also includes the rationality conditions useful in pinning down the parameters of strongly regular graphs and quasi-symmetric designs.

Historically, the forerunners of strongly regular graphs are partially balanced designs and association schemes. Partially balanced designs are generalizations of designs in which the simultaneous occurrence of a point-pair in the blocks is determined by the superposed point-graph. The general question in this regard is that of the determination of suitable conditions under which a strongly

regular graph gives rise to a strongly regular subgraph or a partially balanced design as a substructure. Two important classes of strongly regular graphs studied in Chapter IV are the Mesner family of graphs and the (block) graphs of certain generalized quadrangles. Included among other things are the Bose-Connor property of semiregular group divisible designs, results on special partially balanced designs, and an eigenvalue characterization of partial geometric designs. The theme developed in Chapter IV is further continued in Chapter V, where a detailed discussion of the recent Haemers-Higman study of strongly regular decompositions is made. Included in that chapter are the Hoffman and Cvetcovic coclique bounds and the interlacing theorems.

The construction problem of designs was first handled by statisticians since designs with the right parameters were needed in the design of experiments (see, for example, the book of Raghavarao on designs). However, designs with an aesthetic appeal and elegance of construction are obtained by construction from groups; the constructions of Witt designs from Mathieu groups, in our opinion, bear the best testimony to this fact. A combinatorial exposition of Witt designs was given by Lüneberg in the late 1960's and Chapter VI is devoted to the constructions and combinatorial properties of Witt designs. There are two other independent reasons for the inclusion of Witt designs in this monograph. First, these designs give examples of quasi-symmetric designs and strongly regular graphs (such as the Higman-Sims, Hoffman-Singleton and McLaughlin graphs). In this connection, it should also be pointed out that substantial work in

design theory today concerns itself with various characterizations of Witt designs. Secondly, the Witt designs on 24 and 12 points give examples of Steiner systems $S(t, k, v)$ with $t = 5$, and no non-trivial Steiner system with $t \geq 6$ seems to be known. Many expositions of Witt designs are available in the literature and a similar treatment was given by van Lint. Our treatment in Chapter VI is self-contained and can be understood with no knowledge of group theory.

While the symmetric designs by themselves are improper quasi-symmetric, the 3-designs obtained as extensions of symmetric designs are proper quasi-symmetric, with intersection number pair $(x, y) = (0, y)$. In fact, an observation of Cameron states that the condition $(x, y) = (0, y)$ for a q.s. 3-design characterizes it as an extension of a symmetric design. In his celebrated theorem, Cameron uses the preceding statement to classify the parameters of all the symmetric designs that can be possibly extended. Much recent activity in the area of quasi-symmetric designs is an outcome of Cameron's theorem. Barring the infinite class of Hadamard 3-designs (whose existence is equivalent to the existence of a Hadamard matrix of the corresponding order and whose block graph is just a 1-factor of the complete graph) and two other sporadic examples (one of which is an extension of a projective plane of order ten shown not to exist by a result of Lam and others), the classification theorem of Cameron also includes a putative family of 3-designs, the first object of which is the Witt design on 22 points. In spite of a recent result of Bagchi, it seems difficult to determine whether or not these 3-designs actually exist (particularly since the derived designs themselves are unknown for $\lambda \geq 3$). Chapter VII deals

with this infinite family of 3-designs and their residuals. Among other things, we include a proof to show that a symmetric design is a block residual if and only if its dual is also a block residual.

Chapter VIII is devoted to (general) quasi-symmetric 2-designs and is an important chapter. Observe that most of the 3-designs occurring in Cameron's extension theorem, mentioned in the previous paragraph, are actually triangle-free, i.e., have no three mutually disjoint blocks. Though the concept of a quasi-symmetric design dates back to the late 1960's, it was perhaps only in the early 1980's that the structural investigations of quasi-symmetric designs began with the introduction of a polynomial tool for the study of triangle-free q.s. designs (with $(x, y) = (0, y)$). A large part of Chapter VIII is devoted to results that exploit this polynomial approach. An outstanding conjecture of M. Hall Jr. states that for a fixed $\lambda \geq 2$ there are finitely many symmetric (v, k, λ) -designs. Among other results, Chapter VIII includes a proof of the following analogue of this conjecture: For a fixed $\lambda \geq 2$, there are finitely many proper q.s. (v, k, λ) -designs with the smaller intersection number 0.

Are there t -designs with $t \geq 3$ that are also quasi-symmetric? Cameron showed that non-triviality implies $t \leq 4$. Ito and others proved that up to complementation the only q.s. 4-design is the Witt design $S(4, 7, 23)$. This theme is a particular situation of the Ray-Chaudhuri and Wilson result on tight t -designs. A quadratic with coefficients in the design parameters v, k and λ and whose zeros are the two intersection numbers x and y of a q.s. 3-design is implicit in the

work of Delsarte. A somewhat explicit formulation of this polynomial has been particularly useful in recent investigations. Chapter IX includes a short proof of a recent result which shows that a non-trivial q.s. 3-design with the smaller intersection number one must be the $S(4, 7, 23)$ or its residual. This chapter also includes a conjecture on q.s. 3-designs and an account of some results which indicate a support of that conjecture. If proved, the conjecture will give a result much stronger than that of Ito and others on q.s. 4-designs.

As we already remarked, the recent use of tools from other areas in the theory of quasi-symmetric designs has proved fruitful. The work of Calderbank and Tonchev uses coding theory and other branches and has succeeded in giving many non-existence results for q.s. designs. The last Chapter X is a brief description of these results. Included in Chapter VII is a table of possible parameters of q.s. 2-designs (with small parameters) with certain additional conditions prepared by Neumaier. Neumaier's table seems to have been a motivation for some non-existence results mentioned in the preceding paragraph. We would, however, like to point out that exhaustive tables of q.s. designs will certainly be useful. No such elaborate tables are available at present.

While the last two decades have witnessed some interesting activity in the subject matter, there has not been a single reference dealing exclusively with the topics covered. Ours is an attempt to fill that void. We hope that this monograph will be useful not only to research workers but also to beginning graduate students who would like to gain some acquaintance with the area.

ACKNOWLEDGMENTS

This monograph is an outgrowth of a project spread over several years. In the academic year 1984-85, SSS was at Central Michigan University as a Visiting Research Professor. During this period the authors collaborated on several problems on quasi-symmetric designs. In late 1987 they decided to write a monograph on this topic. MSS would like to acknowledge a Central Michigan University Research Professorship in the Fall of 1988, during which a major portion of the initial draft of this monograph was written. In the academic year 1989-90, SSS was again visiting Central Michigan University. During this time, he was able to work on the manuscript and both the authors completed the final version. SSS gratefully acknowledges the financial support and hospitality provided by Central Michigan University during his visits. He also thanks the University of Bombay for granting him leave for these visits. Finally, both of us would like to thank Barbara Curtiss at Central Michigan University for her splendid job of typing this material.

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