I. BASIC RESULTS FROM DESIGNS

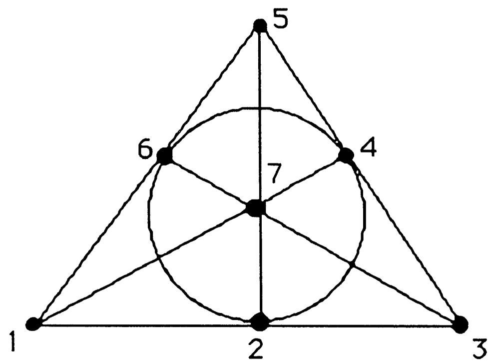
In this first chapter, we collect together and review some basic definitions, notation, and results from design theory. All of these are needed later on. Further details or proofs not given here may be found, for example, in Beth, Jungnickel and Lenz [15], Dembowski [61], Hall [74], Hughes and Piper [95] , or Wallis [177]. We mention also the monographs of Cameron and van Lint [49], Biggs and White [18], and the very recent one by Tonchev [175].

Let $X = \{x_1, x_2, \ldots , x_v\}$ be a finite set of elements called points or treatments and $\beta = \{B_1, B_2, \ldots , B_b\}$ be a finite family of distinct k-subsets of X called blocks. Then the pair $D = (X, \beta)$ is called a t-(v, k, λ) design if every t-subset of X occurs in exactly λ blocks. The integers v, k, and λ are called the parameters of the t-design D. The family consisting of all k-subsets of X forms a k-$(v, k, 1)$ design which is called a complete design. The trivial design is the v-$(v, v, 1)$ design. In order to exclude these degenerate cases we assume always that $v > k > t \geq 1$ and $\lambda \geq 1$. We use the term finite incidence structure to denote a pair (X, β), where X is a finite set and β is a finite family of not necessarily distinct subsets of X. In most of the situations of interest in the later chapters, however, we will have to tighten these restrictions further. For example, though we do not impose the condition that the blocks be distinct sets, that usually would be the case in view of some other stipulations.

A t-design, or more generally an incidence structure, is completely specified up to labellings of its points and blocks by its usual (0, 1)-incidence matrix N. This matrix $N = (n_{ij})_{v \times b}$ is defined by n_{ij} = 1 or 0 according as $x_i \in B_j$ or not. Two designs D_1 and D_2 are said to be isomorphic (denoted by $D_1 \cong D_2$) if there are bijections between their point-sets and block-sets respectively which preserve the incidence. Equivalently, the incidence matrix N_1 of D_1 can be changed to N_2 of D_2 by permuting rows and columns.

We give below three well known and small examples.

Example 1.1. The following picture is a 2-(7, 3, 1) design called the Fano plane. Here the blocks are triples of points which lie on a line or circle.



Example 1.2. Take complements of blocks in Example 1.1. We obtain a 2-(7, 4, 2) design.

Example 1.3. In Example 1.1, add a new symbol ∞ to the point set. Form new blocks by taking the complements of old blocks; in addition take old blocks adjoined with ∞. This new design is a 3-(8, 4, 1) which is an example of the "smallest Hadamard 3-design." The general construction is mentioned later on.

The following simple observation is an important tool in combinatorics. It is known as the method of two way counting and is

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so commonly used in design theory that we often use phrases such as "two way counting produces" or "counting S in two ways gives" etc.

Lemma 1.4. Let U and V be finite sets and let $S \subseteq U \times V = \{(u,v) : u \in U, v \in V\}$. For all $a \in U, b \in V$, define subsets of S by

$$S(a,.) = \{(u,v) \in S : u = a\} \text{ and } S(.,b) = \{(u,v) \in S : v = b\}$$

Then $|S| = \sum_{a \in U} |S(a,.)| = \sum_{b \in V} |S(.,b)|$.

As an immediate application of the above, we have the following result.

Theorem 1.5. Let $D = (X, \beta)$ be a t-(v, k, λ) design. Then the following assertions hold.

(a) For i = 0, 1, ..., t-1, if λ_i denotes the number of blocks containing i points, then λ_i is independent of the choice of i points and in fact, $\lambda_i = \lambda_{i+1} \frac{v-i}{k-i}$.

(b) D is also an i-(v, k, λ_i) design for i = 1, 2, ..., t-1, where

$$\lambda_i = \frac{(v-i)(v-i-1)\cdots(v-t+1)}{(k-i)(k-i-1)\cdots(k-t+1)}\lambda .$$

Proof. (b) follows from (a) using the formula for λ_i in terms of λ_{i+1} and use of induction. Consider (a). We make an induction on j = t-i, i = 0, 1,..., t-1. Assume that any set of i+1 = t-(j-1) points is contained in exactly λ_{i+1} blocks of D. Let $\{x_1, x_2, ..., x_i\}$ be some set of i points of D. Count pairs (x, B) in two ways, where $x \notin \{x_1, x_2, ..., x_i\}$ and $\{x_1, x_2, ..., x_i, x\}$ is contained in B. Then Lemma 1.4 gives $(v-i)\lambda_{i+1} = (k-i)\lambda_i$ which gives λ_i in terms of λ_{i+1} as desired and also shows that λ_i is an

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invariant of D. Observe that our argument has proved both the basis and induction step of our proof.

Remark 1.6. An obvious necessary condition for the existence of a t-(v, k, λ) design is that λ_i be integral for $0 \le i \le t$. This shows, for example, that a 3-(11, 4, 1) design does not exist.

We have in a t-(v, k, λ) design D that $\lambda_0 = b$, the number of blocks, and denote by $\lambda_1 = r$ the number of blocks through any point of D. A t-(v, k, λ) design is also denoted by $\mathbf{S}_\lambda(t, k, v)$. A Steiner system is an $\mathbf{S}_1\,(t, k, v)$. A 2-design (which is not trivial nor complete) is called a balanced incomplete block design (BIBD) or simply a design. For 2-designs, counting flags (i.e., incident point-block pairs) and then 2-flags (i.e., incident pairs of points with blocks) we have

Lemma 1.7. The parameters v, b, r, k, λ of a 2-design satisfy $bk = vr$ and $\lambda(v\text{-}1) = r(k - 1)$.

Remark 1.8. Note that the first relation in Lemma 1.7 holds for any 1-design and both the relations are also immediate consequences of Theorem 1.5 (a) with i = 0 and 1 respectively. Since it sometimes serves our purpose better to list all the parameters, a (v, b, r, k, λ) design is simply a (v, k, λ) design where r and b are given by Lemma 1.7 and must be integers. Then the obvious necessary conditions for the existence of a 2-(v, k, λ) design are $\lambda(v\text{-}1) \equiv 0 \pmod{k\text{-}1}$ and $\lambda v(v\text{-}1) \equiv 0 \pmod{k(k\text{-}1)}$. Hanani [76] has shown these conditions to be sufficient for k = 3, 4, and 5, and every λ, with the exception $v = 15$, $k = 5$, $\lambda = 2$. Wilson [181] has proved the necessary conditions to be sufficient asymptotically. Recently, Teirlinck [168] has proved that non-trivial t-designs without repeated blocks exist for all t. This was a major unsolved problem in design theory.

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Our discussion above should make it sufficiently clear that the existence problem of 2-designs is much harder than the one for 1-designs in view of the following result.

Theorem 1.9. A 1-design (v, b, r, k) without repeated blocks exists if and only if vr=bk and b $\leq \binom{v}{k}$.

We give below D. Billington's [19] elegant proof of this result.

Proof. Observe that a 1-design is just a special case of a $(k, r_1, r_2, ..., r_v)$ - design which consists of a collection of b blocks of cardinality k chosen from a v-set X={1, 2, ..., v}, such that no two blocks are the same, and each $i \in X$ occurs exactly r_i times (i=1, 2, ..., v). A 1-design satisfies $r_1 = r_2 = ... = r_v = r$.

Assertion: Suppose a $(k, r_1, r_2, ..., r_v)$-design exists. If $r_i > r_j$ for some $i \neq j$, then a $(k, r_1, r_2, ..., r_{i-1}, r_i-1, r_{i+1}, ..., r_{j-1}, r_j+1, r_{j+1}, ..., r_v)$-design exists.

To prove the assertion, suppose D is a $(k, r_1, r_2, ..., r_v)$-design with $r_i > r_j$ for some $i \neq j$. Let $B_1, B_2, ..., B_n$ be all the blocks which contain i, and $C_1, C_2, ..., C_m$ be all the blocks which contain j. Since $r_i > r_j$, we necessarily have $0 \leq m < n$. Hence D has a block $B \in \{B_1, B_2, ..., B_n\}$ such that $B \notin \{C_1, C_2, ..., C_m\}$. Form $B^* = (B \setminus \{i\}) \cup \{j\}$. Omitting block B from D and replacing it by B^* results in a $(k, r_1, r_2, ..., r_{i-1}, r_i-1, r_{i+1}, ..., r_{j-1}, r_j+1, r_{j+1}, ..., r_v)$-design. This proves the assertion.

We return to the proof of the theorem. The necessity is obvious. Suppose now vr = bk and b $\leq \binom{v}{k}$. Choose any collection of b distinct k-sets from X = {1, 2, ..., v} obtaining a $(k, r_1, r_2, ..., r_v)$-design D for which $vr = bk = r_1 + r_2 + ... + r_v$. That is, r is the average of $r_1, r_2, ..., r_v$. If $r_i > r_j$ for some $i \neq j$, then using the assertion, we can obtain a $(k, r_1, ..., r_{i-1}, r_i-1, r_i+1, ..., r_{j-1}, r_j+1, ..., r_v)$-design D' for which the average of the replication numbers is still r. Successive use of the assertion results in a (k, r, r, ..., r)-design. This completes the proof of the theorem.

The next result is easily proved.

Lemma 1.10. The incidence matrix N of a 2-(v, k, λ) design satisfies

(i) $NN^t = (r - \lambda)I + \lambda J$, where I is the identity matrix of order v and J the all one matrix.

(ii) $\det(NN^t) = (r - \lambda)^{v-1} rk$.

We now can get Fisher's inequality for 2-designs.

Theorem 1.11. In a non-trivial 2-(v, k, λ) design, the number of blocks b is greater than or equal to the number of points v.

Proof. Here $\det(NN^t) \neq 0$. Thus $v = \text{rank}(NN^t) \leq \text{rank}(N) \leq b$.

A 2-(v, k, λ) design is called symmetric if $v = b$. The next result gives various characterizations of symmetric designs.

Theorem 1.12. The following are equivalent for a 2-(v, k, λ) design D.

(i) D is symmetric.

(ii) $r = k$.

(iii) Any two blocks intersect in λ points.

(iv) The dual D^t of D is a 2-design.

Proof. (Outline) From the proof of Theorem 1.11, v = b implies

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that N is non-singular and by Lemma 1.7, r = k. Since $NN^t = (r-\lambda)I + \lambda J = (k-\lambda)I + \lambda J$ and $NJ = kJ$ we have $N^{-1}J = k^{-1}J$, which can be substituted in $N^tN = N^{-1}NN^tN$ to obtain $N^tN = (k-\lambda)I + \lambda J = NN^t$. This proves that (i) and (ii) are equivalent and imply (iii), which clearly implies (iv). If (iv) holds, then Theorem 1.10 applied to D^t gives $b \leq v$ and $v \leq b$. So (i) holds.

We recall that the dual D^t of a design D is obtained by interchanging the roles of points and blocks. Furthermore, a point B_j is incident with block x_i in D^t if x_i is incident with B_j in D.

Exercise 1.13. Give a matrix-free proof of Theorems 1.10 and 1.11. In fact the following stronger result (Beth et al. [15] or Tonchev [175]) can be proved.

Theorem 1.14. If D is a design with s repeated blocks, then $b \geq sv$.

From Lemma 1.10, if D is symmetric then $(\det(N))^2 = \det(NN^t) = (k-\lambda)^{v-1}k^2$. Since all the numbers involved are integers, we obtain the following result proved independently by Schutzenberger [142], Chowla and Ryser [55], and S.S. Shrikhande [157].

Theorem 1.15. If there exists a symmetric 2-(v, k, λ) design with v even, then $(k - \lambda)$ must be a square.

The above theorem rules out, for example, a symmetric 2-(22, 7, 2) design. The case of odd v (Theorem 1.16) needs deeper number theoretic arguments than the simple matrix proof of Theorem 1.15. Theorem 1.16 was proved by Chowla and Ryser [55] (a proof based on the calculation of the Hasse-Minkowski invariant was also given by

S.S. Shrikhande [157]) and Theorems 1.15 and 1.16 are together now known by the name Bruck-Ryser-Chowla Theorem.

Theorem 1.16. A necessary condition for the existence of a symmetric 2-(v, k, λ) design with odd v is that the equation $x^2 = (k - \lambda) y^2 + (-1)^{(v-1)/2} \lambda z^2$ has an integral solution $(x, y, z) \neq (0, 0, 0)$.

The above result rules out, for example, a symmetric 2-(43, 7, 1) design. It was not known whether the conditions of the Bruck-Ryser-Chowla Theorem were sufficient. Apparently they are not, in view of a recent paper by Lam et al. [102].

Given a symmetric 2-(v, k, λ) design D and a block B, we can obtain a 2-design D^B called the residual of D with respect to B. The points of D^B are the points of D outside B. The blocks of D^B are the blocks of D minus the points of B. It is easily checked that D^B is a 2-$(v\text{-}k, k\text{-}\lambda, \lambda)$ design. A 2-design is quasi-residual if it has the right parameters to be the residual of a suitable symmetric 2-design. To be specific a 2-(v, k, λ) design is called quasi-residual if $r = k + \lambda$ (equivalently $v = k(k + \lambda - 1)/\lambda$).

For $\lambda = 1$, any quasi-residual design D is a 2-$(k^2, k, 1)$ design, i.e., an affine plane. The familiar process of embedding an affine plane in a projective plane immediately shows that a quasi-residual design with $\lambda = 1$ is the residual of a unique symmetric 2-design. The following result of Hall and Connor [75] covers the case $\lambda = 2$.

Theorem 1.17. Any quasi-residual 2-$(v, k, 2)$ design is the residual of a unique symmetric 2-design.

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We should also mention that the exact analogue of Theorem 1.17 for $\lambda \geq 3$ is not true in general and there exist counterexamples (e.g. [74]). However Bose, S.S. Shrikhande, and Singhi [30] proved that there exists a function $f(\lambda)$ on the positive integers such that any quasi-residual 2-(v, k, λ) design with $\lambda \geq 3$ is the residual of a unique symmetric 2-design if $k > f(\lambda)$.

Let D be a t-(v, k, λ) design and p a point of D. The derived design (or the point contraction) D_p with respect to p is the $(t - 1)$-$(v - 1, k - 1, \lambda)$ design whose points are the points of D other than p and whose blocks are the blocks of D passing through p. The residual design D^p is a $(t - 1)$-$(v - 1, k, \lambda_{t-1} - \lambda)$ design, whose point set is that of D_p and whose blocks are those blocks of D missing p. A t-design D is called extendable if there exists a $(t+1)$-design D^* and a point ∞ of D^* such that $D \cong D^*_\infty$. If D is a given t-(v, k, λ) design, then all the derived designs of D are $(t-1)$-$(v-1, k-1, \lambda)$ designs and therefore have identical parameter sets. This simple observation gives as a special case

Lemma 1.18. If D is a 3-design and if for some point p the derived design D_p is a symmetric design, then every derived design D_q of D is a symmetric design with the same parameters as those of D_p.

Proof. Apply the definition of a symmetric design and observe that D_p and D_q have identical parameters.

The naive looking Lemma 1.18 is an important tool in the proof of Cameron's theorem discussed at the end of this chapter. The next result of Hughes [92] gives a simple necessary condition for extendability.

Proposition 1.19. If a t-(v, k, λ) design with b blocks is extendable, then $k + 1$ divides $b(v + 1)$.

Proof. Let D be a t-(v, k, λ) design and D^* its extension. Apply the condition $bk = vr$ to D^* and note that here the number of blocks through a point of D^* is the number of blocks of D.

Applying Proposition 1.19 to a projective plane of order n (= symmetric 2-$(n^2 + n + 1, n + 1, 1)$ design), Hughes [92] obtained the following:

Theorem 1.20. If a projective plane of order n is extendable, then $n = 2, 4$, or 10.

Remarks 1.21. The projective plane of order 2 is uniquely extendable to a 3-(8, 4, 1) design. The projective plane of order 4 is three times extendable, giving the famous and unique 3-(22, 6, 1), 4-(23, 7, 1), and 5-(24, 8, 1) designs (see e.g. [15]). Lam et al. [101], using a computer, have shown that a plane of order ten is not extendable. (Lam, Thiel, and Swiercz have apparently, very recently, ruled out the existence of a plane of order ten [102]).

We now recall some well known and important constructions for designs. Some of these will be often referred to later on. We list these facts as remarks, and for details refer to the references mentioned earlier.