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Geometry of Sporadic Groups I
Petersen and tilde geometries

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0521413621 - Geometry of Sporadic Groups I Petersen and Tilde Geometries

A. A. Ivanov

Frontmatter

[More information](#)

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

- 4 W. Miller, Jr. *Symmetry and separation of variables*
- 6 H. Minc *Permanents*
- 11 W. B. Jones and W. J. Thron *Continued fractions*
- 12 N. F. G. Martin and J. W. England *Mathematical theory of entropy*
- 18 H. O. Fattorini *The Cauchy problem*
- 19 G. G. Lorentz, K. Jetter and S. D. Riemenschneider *Birkhoff interpolation*
- 21 W. T. Tutte *Graph theory*
- 22 J. R. Bastida *Field extensions and Galois theory*
- 23 J. R. Cannon *The one-dimensional heat equation*
- 25 A. Salomaa *Computation and automata*
- 26 N. White (ed.) *Theory of matroids*
- 27 N. H. Bingham, C. M. Goldie and J. L. Teugels *Regular variation*
- 28 P. P. Petrushev and V. A. Popov *Rational approximation of real functions*
- 29 N. White (ed.) *Combinatorial geometries*
- 30 M. Pohst and H. Zassenhaus *Algorithmic algebraic number theory*
- 31 J. Aczel and J. Dhombres *Functional equations containing several variables*
- 32 M. Kuczma, B. Choczewski and R. Ger *Iterative functional equations*
- 33 R. V. Ambartzumian *Factorization calculus and geometric probability*
- 34 G. Gripenberg, S.-O. Londen and O. Staffans *Volterra integral and functional equations*
- 35 G. Gasper and M. Rahman *Basic hypergeometric series*
- 36 E. Torgersen *Comparison of statistical experiments*
- 37 A. Neumaier *Intervals methods for systems of equations*
- 38 N. Korneichuk *Exact constants in approximation theory*
- 39 R. A. Brualdi and H. J. Ryser *Combinatorial matrix theory*
- 40 N. White (ed.) *Matroid applications*
- 41 S. Sakai *Operator algebras in dynamical systems*
- 42 W. Hodges *Model theory*
- 43 H. Stahl and V. Totik *General orthogonal polynomials*
- 44 R. Schneider *Convex bodies*
- 45 G. Da Prato and J. Zabczyk *Stochastic equations in infinite dimensions*
- 46 A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. Ziegler *Oriented matroids*
- 47 E. A. Edgar and L. Sucheston *Stopping times and directed processes*
- 48 C. Sims *Computation with finitely presented groups*
- 49 T. Palmer *Banach algebras and the general theory of *-algebras*
- 50 F. Borceux *Handbook of categorical algebra I*
- 51 F. Borceux *Handbook of categorical algebra II*
- 52 F. Borceux *Handbook of categorical algebra III*
- 54 A. Katok and B. Hassleblatt *Introduction to the modern theory of dynamical systems*
- 55 V. N. Sachkov *Combinatorial methods in discrete mathematics*
- 56 V. N. Sachkov *Probabilistic methods in discrete mathematics*
- 57 P. M. Cohn *Skew Fields*
- 58 Richard J. Gardner *Geometric tomography*
- 59 George A. Baker, Jr. and Peter Graves-Morris *Padé approximants*
- 60 Jan Krajčec *Bounded arithmetic, propositional logic, and complex theory*
- 61 H. Gromer *Geometric applications of Fourier series and spherical harmonics*
- 62 H. O. Fattorini *Infinite dimensional optimization and control theory*
- 63 A. C. Thompson *Minkowski geometry*
- 64 R. B. Bapat and T. E. S. Raghavan *Nonnegative matrices and applications*
- 65 K. Engel *Sperner theory*
- 66 D. Cvetkovic, P. Rowlinson and S. Simic *Eigenspaces of graphs*
- 67 F. Bergeron, G. Labelle and P. Leroux *Combinatorial species and tree-like structures*
- 68 R. Goodman and N. Wallach *Representations of the classical groups*
- 70 A. Pietsch and J. Wenzel *Orthonormal systems and Banach space geometry*

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Imperial College, London



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Contents

	<i>page</i>	<i>ix</i>
<i>Preface</i>		
1 Introduction		1
1.1 Basic definitions		2
1.2 Morphisms of geometries		5
1.3 Amalgams		7
1.4 Geometrical amalgams		9
1.5 Universal completions and covers		10
1.6 Tits geometries		11
1.7 Alt_7 -geometry		16
1.8 Symplectic geometries over $GF(2)$		17
1.9 From classical to sporadic geometries		19
1.10 The main results		21
1.11 Representations of geometries		23
1.12 The stages of classification		26
1.13 Consequences and development		33
1.14 Terminology and notation		42
2 Mathieu groups		49
2.1 The Golay code		50
2.2 Constructing a Golay code		51
2.3 The Steiner system $S(5, 8, 24)$		53
2.4 Linear groups		56
2.5 The quad of order (2, 2)		59
2.6 The rank 2 T -geometry		62
2.7 The projective plane of order 4		64
2.8 Uniqueness of $S(5, 8, 24)$		71
2.9 Large Mathieu groups		74
2.10 Some further subgroups of Mat_{24}		76

vi	<i>Contents</i>	
2.11	Little Mathieu groups	81
2.12	Fixed points of a 3-element	85
2.13	Some odd order subgroups in Mat_{24}	87
2.14	Involutions in Mat_{24}	90
2.15	Golay code and Todd modules	95
2.16	The quad of order (3,9)	97
3	Geometry of Mathieu groups	100
3.1	Extensions of planes	101
3.2	Maximal parabolic geometry of Mat_{24}	102
3.3	Minimal parabolic geometry of Mat_{24}	106
3.4	Petersen geometries of the Mathieu groups	112
3.5	The universal cover of $\mathcal{G}(Mat_{22})$	117
3.6	$\mathcal{G}(Mat_{23})$ is 2-simply connected	122
3.7	Diagrams for $\mathcal{H}(Mat_{24})$	124
3.8	More on Golay code and Todd modules	130
3.9	Diagrams for $\mathcal{H}(Mat_{22})$	132
3.10	Actions on the sextets	138
4	Conway groups	141
4.1	Lattices and codes	141
4.2	Some automorphisms of lattices	147
4.3	The uniqueness of the Leech lattice	150
4.4	Coordinates for Leech vectors	153
4.5	Co_1 , Co_2 and Co_3	158
4.6	The action of Co_1 on $\bar{\Lambda}_4$	160
4.7	The Leech graph	163
4.8	The centralizer of an involution	169
4.9	Geometries of Co_1 and Co_2	173
4.10	The affine Leech graph	178
4.11	The diagram of Δ	189
4.12	The simple connectedness of $\mathcal{G}(Co_2)$ and $\mathcal{G}(Co_1)$	193
4.13	McL geometry	198
4.14	Geometries of $3 \cdot U_4(3)$	203
5	The Monster	210
5.1	Basic properties	211
5.2	The tilde geometry of the Monster	216
5.3	The maximal parabolic geometry	218
5.4	Towards the Baby Monster	222
5.5	${}^2E_6(2)$ -subgeometry	224
5.6	Towards the Fischer group $M(24)$	227

Contents

vii

5.7	Identifying $M(24)$	231
5.8	Fischer groups and their properties	236
5.9	Geometry of the Held group	242
5.10	The Baby Monster graph	244
5.11	The simple connectedness of $\mathcal{G}(BM)$	256
5.12	The second Monster graph	259
5.13	Uniqueness of the Monster amalgam	265
5.14	On existence and uniqueness of the Monster	268
5.15	The simple connectedness of $\mathcal{G}(M)$	271
6	From C_n - to T_n -geometries	272
6.1	On induced modules	273
6.2	A characterization of $\mathcal{G}(3 \cdot Sp_4(2))$	276
6.3	Dual polar graphs	280
6.4	Embedding the symplectic amalgam	285
6.5	Constructing T -geometries	288
6.6	The rank 3 case	290
6.7	Identification of $J(n)$	293
6.8	A special class of subgroups in $J(n)$	295
6.9	The $\mathcal{J}(n)$ are 2-simply connected	297
6.10	A characterization of $\mathcal{J}(n)$	301
6.11	No tilde analogues of the Alt_7 -geometry	303
7	2-Covers of P -geometries	307
7.1	On P -geometries	307
7.2	A sufficient condition	313
7.3	Non-split extensions	315
7.4	$\mathcal{G}(3^{23} \cdot Co_2)$	318
7.5	The rank 5 case: bounding the kernel	321
7.6	$\mathcal{G}(3^{4371} \cdot BM)$	327
7.7	Some further s -coverings	330
8	Y -groups	332
8.1	Some history	333
8.2	The 26-node theorem	335
8.3	From Y -groups to Y -graphs	337
8.4	Some orthogonal groups	340
8.5	Fischer groups as Y -groups	345
8.6	The monsters	351
9	Locally projective graphs	358
9.1	Groups acting on graphs	359

Cambridge University Press

0521413621 - Geometry of Sporadic Groups I Petersen and Tilde Geometries

A. A. Ivanov

Frontmatter

[More information](#)

viii

Contents

9.2	Classical examples	362
9.3	Locally projective lines	367
9.4	Main types	370
9.5	Geometrical subgraphs	374
9.6	Further properties of geometrical subgraphs	379
9.7	The structure of P	383
9.8	Complete families of geometrical subgraphs	386
9.9	Graphs of small girth	389
9.10	Projective geometries	392
9.11	Petersen geometries	394
	<i>Bibliography</i>	398
	<i>Index</i>	406

Preface

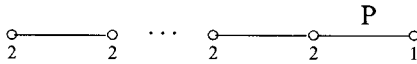
Sporadic simple groups are the most fascinating objects in modern algebra. The discovery of these groups and especially of the Monster is considered to be one of the most important contributions of the classification of finite simple groups to mathematics. Some of the sporadic simple groups were originally realized as automorphism groups of certain combinatorial–geometrical structures like Steiner systems, distance-regular graphs, Fischer spaces *etc.*, but it was the epoch-making paper [Bue79] by F. Buekenhout which brought an axiomatic foundation for these and related structures under the name “diagram geometries”. Buildings of finite groups of Lie type form a special class of diagram geometries known as Tits geometries. This gives a hope that diagram geometries might serve as a background for a uniform treatment of all finite simple groups.

If G is a finite group of Lie type in characteristic p , then its Tits geometry $\mathcal{G}(G)$ can be constructed as the coset geometry with respect to the maximal parabolic subgroups which are maximal overgroups of the normalizer in G of a Sylow p -subgroup (this normalizer is known as the Borel subgroup). Thus $\mathcal{G}(G)$ can be defined in abstract group-theoretical terms. Similar abstract construction applied to sporadic simple groups led to maximal [RSm80] and minimal [RSt84] parabolic geometries, most naturally associated with the sporadic simple groups. Notice that besides the parabolic geometries there are a number of other nice diagram geometries associated with sporadic groups.

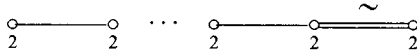
Tits geometries are characterized by the property that all their rank 2 residues are generalized polygons. Geometries of sporadic groups besides the generalized polygons involve c -geometries (which are geometries of vertices and edges of complete graphs), the geometry of the Petersen

graph, tilde geometry (a triple cover of the generalized quadrangle of order (2,2)) and a few other rank 2 residues.

In the mid 80's the classification project of finite Tits geometries attracted a lot of interest, motivated particularly by the revision program of the classification of finite simple groups (see [Tim84]). It was natural to extend this project to geometries of sporadic groups and to try to characterize such geometries by their diagrams. For two classes of diagrams, namely



and



the complete classification under the flag-transitivity assumption was achieved by S.V. Shpectorov and the author of the present volume [ISh94b]. Geometries with the above diagrams are called, respectively, Petersen and tilde geometries. A complete self-contained exposition of the classification of flag-transitive Petersen and tilde geometries is the main goal of the two volume monograph of which the present is the first volume.

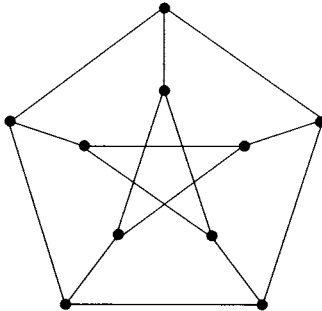
To provide the reader with an idea what sporadic group geometries look like we present the axioms for the smallest case.

A Petersen geometry of rank 3 is a 3-partite graph \mathcal{G} with the partition

$$\mathcal{G} = \mathcal{G}^1 \cup \mathcal{G}^2 \cup \mathcal{G}^3$$

which possesses the following properties. For a vertex $x \in \mathcal{G}$ let $\text{res}(x)$ denote the subgraph in \mathcal{G} induced on the set of vertices adjacent to x . For $x_i \in \mathcal{G}^i$, $1 \leq i \leq 3$, the following hold:

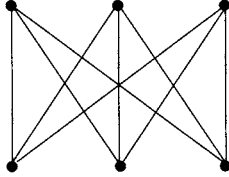
$\text{res}(x_1)$ is the incidence graph of vertices and edges of the Petersen graph



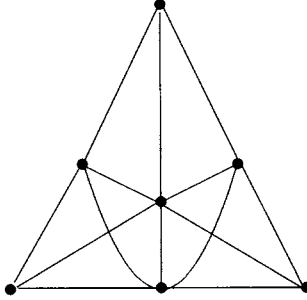
Preface

xi

$\text{res}(x_2)$ is the complete bipartite graph $K_{3,3}$



$\text{res}(x_3)$ is the incidence graph of seven points and seven lines of the Fano plane



The geometry \mathcal{G} as above is flag-transitive if its automorphism group acts transitively on the set of maximal complete subgraphs (such a subgraph contains one vertex from each part).

It was shown by S.V. Shpectorov in [Sh85] that there exist exactly two flag-transitive Petersen geometries of rank 3. Their automorphism groups are isomorphic respectively to the automorphism group $\text{Aut } \text{Mat}_{22}$ of the Mathieu group Mat_{22} and to a non-split extension of $\text{Aut } \text{Mat}_{22}$ by a subgroup of order 3. This was the first step in the classification project of Petersen and tilde geometries.

Our strategy of classification, first implemented in [Sh85], is based on analysis of amalgams of maximal parabolic subgroups and calculation of the universal covers and consists of two principal and rather independent steps.

Step 1. To describe all known pairs (\mathcal{G}, G) where \mathcal{G} is a Petersen or tilde geometry and G is a flag-transitive automorphism group of \mathcal{G} , calculate the universal cover of \mathcal{G} and determine its flag-transitive quotients.

Step 2. To show that the amalgam of maximal parabolic subgroups, corresponding to a flag-transitive action on a Petersen or tilde geometry

\mathcal{H} , is isomorphic to such an amalgam corresponding to a pair (\mathcal{G}, G) described in Step 1. By a standard principle this means that \mathcal{H} is a flag-transitive quotient of the universal cover of \mathcal{G} .

The main goal of the present volume is to realize Step 1. The local analysis of amalgams needed for Step 2 will be given in the second volume. Here we also discuss various applications and implications of the classification of flag-transitive Petersen and tilde geometries.

In Chapter 1 we start with a review of the main notions and principles concerning the diagram geometries and their flag-transitive automorphism groups. Then we formulate and discuss the results of the classification project for flag-transitive Petersen and tilde geometries. In Chapter 2 we prove the existence and uniqueness of the (binary) Golay code and the Steiner system $S(5, 8, 24)$. Our approach is a mixture of the approach of Conway (in [Con71]) who constructs the Golay code as the quadratic residue code over $GF(23)$ and the approach in Lüneburg (in [Lün69]) who treats the Steiner system $S(5, 8, 24)$ as an extension of the projective plane of order 4. The approach provides us with a strong background to define the Mathieu groups and to study their subgroup structure. In Chapter 3 we define and study geometries of the Mathieu groups. We refer to computer calculations performed independently in [Hei91] and [ISh89a] to claim the simple connectedness of the tilde geometry $\mathcal{G}(Mat_{24})$. The simple connectedness proofs for the Petersen geometries $\mathcal{G}(Mat_{22})$ and $\mathcal{G}(Mat_{23})$ which we present here are basically the original ones from [Sh85] and [ISh90a]. In Chapter 4 we follow [Con69] and [KKM91] to establish the existence and uniqueness of the Leech lattice. This approach immediately gives the order and basic properties of the automorphism group of the Leech lattice. We present a detailed study of the action of Co_1 on $\bar{\Lambda}_4$ and of an orbital graph associated with this action. This graph is the collinearity graph of the tilde geometry $\mathcal{G}(Co_1)$. We present the simple connectedness proofs for $\mathcal{G}(Co_2)$ and $\mathcal{G}(Co_1)$, originally given in [Sh92] and [Iv92a], respectively. At the end of Chapter 4 we discuss geometries of certain subgroups in the Conway group Co_1 . In Chapter 5 we prove the simple connectedness of the tilde geometry $\mathcal{G}(M)$ of the Monster. We start with an amalgam \mathcal{M} similar to the amalgam of maximal parabolics associated with the action of the Monster on its tilde geometry and consider a faithful completion G of \mathcal{M} . We define a number of subgroups in G associated with certain subgeometries in $\mathcal{G}(M)$. Applying the simple connectedness of these subgeometries originally established in [Iv92c], [Iv94] and [Iv95]

Preface

xiii

we identify in G the subgroups $3 \cdot M(24)$ and $2 \cdot BM$. By considering the subgeometry in $\mathcal{G}(M)$ formed by the fixed points of an element of order 7, we construct the tilde geometry $\mathcal{G}(He)$ of the Held group. We define a graph Γ on the set of Baby Monster subgeometries in $\mathcal{G}(M)$ (called the second Monster graph) and study its local properties. We apply the triangulability of Γ proved in [ASeg92] to establish the simple connectedness of $\mathcal{G}(M)$. In Chapter 6 we follow [ISh93a] to construct an infinite family of tilde geometries associated with some non-split extensions of symplectic groups over $GF(2)$. In the last section of Chapter 6 we follow [ISh90a] to prove the non-existence of tilde analogues of the exceptional C_3 -geometry $\mathcal{G}(Alt_7)$. In Chapter 7 we construct the Petersen geometries associated with the non-split extensions

$$3 \cdot \text{Aut } Mat_{22}, \quad 3^{23} \cdot Co_2, \quad 3^{4371} \cdot BM$$

and prove their 2-simple connectedness following [Sh92] and [ISh93b]. In Chapter 8 we discuss the identification proof of Y_{555} with the Bimonster. In this proof the simple connectedness of $\mathcal{G}(M)$ plays an essential rôle. In Chapter 9 we consider locally projective graphs and show how the classification of the flag-transitive Petersen geometries implies description of a class of locally projective graphs of girth 5. Originally this reduction was proved in [Iv88], [Iv90] (see also a survey [Iv93a]). In this volume we do not treat the fourth Janko group J_4 and its Petersen geometry $\mathcal{G}(J_4)$, and refer the reader to [IMe93] where the group and its geometry are constructed and characterized starting with very basic principles.

I would like to thank S.V. Shpectorov for the fruitful cooperation on the classification project for flag-transitive Petersen and tilde geometries which led to its completion. I am grateful to B. Baumeister, S. Hobart, G. Glauberman, C.E. Praeger, C. Wiedorn who read various parts of preliminary versions of the volume and suggested a number of corrections. I am glad to acknowledge that many suborbit diagrams presented in the volume have been computed by D.V. Pasechnik.