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Volume 76

Geometry of Sporadic Groups I
Petersen and tilde geometries

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Petersen and tilde geometries

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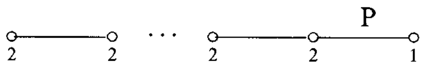
Preface

Sporadic simple groups are the most fascinating objects in modern algebra. The discovery of these groups and especially of the Monster is considered to be one of the most important contributions of the classification of finite simple groups to mathematics. Some of the sporadic simple groups were originally realized as automorphism groups of certain combinatorial–geometrical structures like Steiner systems, distance-regular graphs, Fischer spaces *etc.*, but it was the epoch-making paper [Bue79] by F. Buekenhout which brought an axiomatic foundation for these and related structures under the name "diagram geometries". Buildings of finite groups of Lie type form a special class of diagram geometries known as Tits geometries. This gives a hope that diagram geometries might serve as a background for a uniform treatment of all finite simple groups.

If G is a finite group of Lie type in characteristic p, then its Tits geometry $\mathcal{G}(G)$ can be constructed as the coset geometry with respect to the maximal parabolic subgroups which are maximal overgroups of the normalizer in G of a Sylow p-subgroup (this normalizer is known as the Borel subgroup). Thus $\mathcal{G}(G)$ can be defined in abstract group-theoretical terms. Similar abstract construction applied to sporadic simple groups led to maximal [RSm80] and minimal [RSt84] parabolic geometries, most naturally associated with the sporadic simple groups. Notice that besides the parabolic geometries there are a number of other nice diagram geometries associated with sporadic groups.

Tits geometries are characterized by the property that all their rank 2 residues are generalized polygons. Geometries of sporadic groups besides the generalized polygons involve c-geometries (which are geometries of vertices and edges of complete graphs), the geometry of the Petersen graph, tilde geometry (a triple cover of the generalized quadrangle of order (2,2)) and a few other rank 2 residues.

In the mid 80's the classification project of finite Tits geometries attracted a lot of interest, motivated particularly by the revision program of the classification of finite simple groups (see [Tim84]). It was natural to extend this project to geometries of sporadic groups and to try to characterize such geometries by their diagrams. For two classes of diagrams, namely



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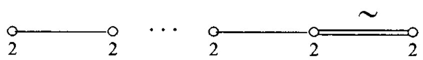


the complete classification under the flag-transitivity assumption was achieved by S.V. Shpectorov and the author of the present volume [ISh94b]. Geometries with the above diagrams are called, respectively, Petersen and tilde geometries. A complete self-contained exposition of the classification of flag-transitive Petersen and tilde geometries is the main goal of the two volume monograph of which the present is the first volume.

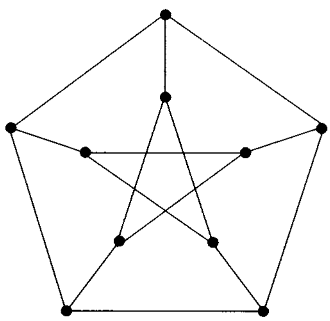
To provide the reader with an idea what sporadic group geometries look like we present the axioms for the smallest case.

A Petersen geometry of rank 3 is a 3-partite graph $\mathcal{G}$ with the partition

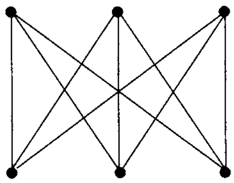
$$\mathcal{G} = \mathcal{G}^1 \cup \mathcal{G}^2 \cup \mathcal{G}^3$$

which possesses the following properties. For a vertex $x \in \mathcal{G}$ let res(x) denote the subgraph in $\mathcal{G}$ induced on the set of vertices adjacent to x. For $x_i \in \mathcal{G}^i$, $1 \le i \le 3$, the following hold:

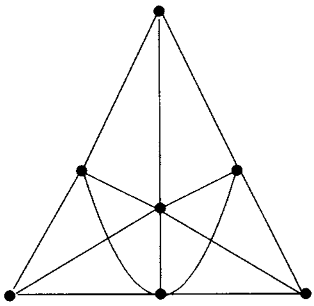
res(x_1) is the incidence graph of vertices and edges of the Petersen graph



res(x_2) is the complete bipartite graph $K_{3,3}$



res(x_3) is the incidence graph of seven points and seven lines of the Fano plane



The geometry $\mathcal{G}$ as above is flag-transitive if its automorphism group acts transitively on the set of maximal complete subgraphs (such a subgraph contains one vertex from each part).

It was shown by S.V. Shpectorov in [Sh85] that there exist exactly two flag-transitive Petersen geometries of rank 3. Their automorphism groups are isomorphic respectively to the automorphism group Aut Mat_{22} of the Mathieu group Mat_{22} and to a non-split extension of Aut Mat_{22} by a subgroup of order 3. This was the first step in the classification project of Petersen and tilde geometries.

Our strategy of classification, first implemented in [Sh85], is based on analysis of amalgams of maximal parabolic subgroups and calculation of the universal covers and consists of two principal and rather independent steps.

Step 1. To describe all known pairs $(\mathcal{G}, G)$ where $\mathcal{G}$ is a Petersen or tilde geometry and G is a flag-transitive automorphism group of $\mathcal{G}$, calculate the universal cover of $\mathcal{G}$ and determine its flag-transitive quotients.

Step 2. To show that the amalgam of maximal parabolic subgroups, corresponding to a flag-transitive action on a Petersen or tilde geometry

$\mathcal{H}$, is isomorphic to such an amalgam corresponding to a pair $(\mathcal{G}, G)$ described in Step 1. By a standard principle this means that $\mathcal{H}$ is a flag-transitive quotient of the universal cover of $\mathcal{G}$.

The main goal of the present volume is to realize Step 1. The local analysis of amalgams needed for Step 2 will be given in the second volume. Here we also discuss various applications and implications of the classification of flag-transitive Petersen and tilde geometries.

In Chapter 1 we start with a review of the main notions and principles concerning the diagram geometries and their flag-transitive automorphism groups. Then we formulate and discuss the results of the classification project for flag-transitive Petersen and tilde geometries. In Chapter 2 we prove the existence and uniqueness of the (binary) Golay code and the Steiner system $S(5, 8, 24)$. Our approach is a mixture of the approach of Conway (in [Con71]) who constructs the Golay code as the quadratic residue code over $GF(23)$ and the approach in Lüneburg (in [Lün69]) who treats the Steiner system $S(5, 8, 24)$ as an extension of the projective plane of order 4. The approach provides us with a strong background to define the Mathieu groups and to study their subgroup structure. In Chapter 3 we define and study geometries of the Mathieu groups. We refer to computer calculations performed independently in [Hei91] and [ISh89a] to claim the simple connectedness of the tilde geometry $\mathcal{G}(Mat_{24})$. The simple connectedness proofs for the Petersen geometries $\mathcal{G}(Mat_{22})$ and $\mathcal{G}(Mat_{23})$ which we present here are basically the original ones from [Sh85] and [ISh90a]. In Chapter 4 we follow [Con69] and [KKM91] to establish the existence and uniqueness of the Leech lattice. This approach immediately gives the order and basic properties of the automorphism group of the Leech lattice. We present a detailed study of the action of Co_1 on $\bar{\Lambda}_4$ and of an orbital graph associated with this action. This graph is the collinearity graph of the tilde geometry $\mathcal{G}(Co_1)$. We present the simple connectedness proofs for $\mathcal{G}(Co_2)$ and $\mathcal{G}(Co_1)$, originally given in [Sh92] and [Iv92a], respectively. At the end of Chapter 4 we discuss geometries of certain subgroups in the Conway group Co_1. In Chapter 5 we prove the simple connectedness of the tilde geometry $\mathcal{G}(M)$ of the Monster. We start with an amalgam $\mathcal{M}$ similar to the amalgam of maximal parabolics associated with the action of the Monster on its tilde geometry and consider a faithful completion G of $\mathcal{M}$. We define a number of subgroups in G associated with certain subgeometries in $\mathcal{G}(M)$. Applying the simple connectedness of these subgeometries originally established in [Iv92c], [Iv94] and [Iv95] we identify in G the subgroups $3 \cdot M(24)$ and $2 \cdot BM$. By considering the subgeometry in $\mathcal{G}(M)$ formed by the fixed points of an element of order 7, we construct the tilde geometry $\mathcal{G}(He)$ of the Held group. We define a graph Γ on the set of Baby Monster subgeometries in $\mathcal{G}(M)$ (called the second Monster graph) and study its local properties. We apply the triangulability of Γ proved in [ASeg92] to establish the simple connectedness of $\mathcal{G}(M)$. In Chapter 6 we follow [ISh93a] to construct an infinite family of tilde geometries associated with some non-split extensions of symplectic groups over $GF(2)$. In the last section of Chapter 6 we follow [ISh90a] to prove the non-existence of tilde analogues of the exceptional C_3-geometry $\mathcal{G}(Alt_7)$. In Chapter 7 we construct the Petersen geometries associated with the non-split extensions

$$3 \cdot \mathrm{Aut}\, Mat_{22}, \quad 3^{23} \cdot Co_2, \quad 3^{4371} \cdot BM$$

and prove their 2-simple connectedness following [Sh92] and [ISh93b]. In Chapter 8 we discuss the identification proof of Y_{555} with the Bimonster. In this proof the simple connectedness of $\mathcal{G}(M)$ plays an essential rôle. In Chapter 9 we consider locally projective graphs and show how the classification of the flag-transitive Petersen geometries implies description of a class of locally projective graphs of girth 5. Originally this reduction was proved in [Iv88], [Iv90] (see also a survey [Iv93a]). In this volume we do not treat the fourth Janko group J_4 and its Petersen geometry $\mathcal{G}(J_4)$, and refer the reader to [IMe93] where the group and its geometry are constructed and characterized starting with very basic principles.

I would like to thank S.V. Shpectorov for the fruitful cooperation on the classification project for flag-transitive Petersen and tilde geometries which led to its completion. I am grateful to B. Baumeister, S. Hobart, G. Glauberman, C.E. Praeger, C. Wiedorn who read various parts of preliminary versions of the volume and suggested a number of corrections. I am glad to acknowledge that many suborbit diagrams presented in the volume have been computed by D.V. Pasechnik.