1
Introduction

In this introductory chapter we collect basic definitions, formulate main results and discuss some of the motivations and consequences. In Section 1.1 we start with an informal review of classical geometries in order to motivate the general notion of geometry as introduced by J. Tits in the 50's. In Section 1.2 we discuss morphisms of geometries and two of their most important special cases, coverings and automorphisms. Our main interest is in flag-transitive geometries. By a standard principle a flag-transitive geometry $\mathcal{G}$ can be uniquely reconstructed from its flag-transitive automorphism group G and the embedding in G of the amalgam $\mathcal{A}$ (defined in Section 1.3) of maximal parabolic subgroups corresponding to the action of G on $\mathcal{G}$. In Section 1.4 we formulate a condition under which an abstract group G and a subamalgam $\mathcal{A}$ in G lead to a geometry. In Section 1.5 we formulate the most fundamental principle in the area which relates the universal cover of a flag-transitive geometry $\mathcal{G}$ and the universal completion of the amalgam of maximal parabolic subgroups corresponding to a flag-transitive action on $\mathcal{G}$. In Section 1.6 we discuss parabolic geometries of finite groups of Lie type. These geometries belong to the class of so-called Tits geometries characterized by the property that all rank 2 residues are classical generalized polygons. We formulate the local characterization of Tits geometries which shows a special rôle of C_3-geometries. We also formulate a very useful description of flag-transitive automorphism groups of classical Tits geometries due to G. Seitz. A very important non-classical Tits geometry, known as the Alt_7-geometry, is discussed in Section 1.7. In Section 1.8 we apply the characterization of Tits geometries to $C_n(2)$-geometries which play a very special rôle in our exposition. In Section 1.9 we mimic the construction of $C_n(2)$-geometries of symplectic groups to produce a rank 5 tilde geometry of the Monster group. In Section 1.10 the classification result for flag-transitive Petersen and tilde geometries is stated, which shows in particular that the Monster is strongly characterized as a flag-transitive automorphism group of a rank 5 tilde geometry. In Section 1.11 we introduce and discuss a very important notion of natural representations of geometries. Section 1.12 contains a brief historical essay about the classification of flag-transitive Petersen and tilde geometries. In Section 1.13 we present some implications of the classification including the identification of Y-groups. In the final section of the chapter we fix our terminology and notation concerning groups, graphs and geometries. The terminology and notation are mostly standard and we start using them in the earlier sections of the chapter without explanations.

1.1 Basic definitions

We start this chapter with a brief and informal review of the geometries of classical groups in order to motivate the general definition of geometries.

Let G be a finite classical group (assuming the projective version). The group G itself and its geometry can be defined in terms of the natural module which is an n-dimensional vector space $V = V_n(q)$ over the Galois field $GF(q)$ of order q. Here q is a power of a prime number p called the *characteristic* of the field. There is a sesquilinear form Ψ on V which is either trivial (identically equal to zero) or non-singular and the elements of G are projective transformations of V which preserve Ψ up to multiplication by scalars. If Ψ is trivial then G is just a projective linear group associated with V. If Ψ is non-singular, it is symplectic, unitary or orthogonal and G is the symplectic, unitary or orthogonal group of a suitable type determined by n, q and the type of Ψ. We have introduced the trivial form in the case of linear groups in order to treat all classical groups uniformly.

For a subspace W of V we can consider the restriction of Ψ to W. The subspaces on which Ψ restricts trivially play a very special rôle and they are called *totally singular subspaces* of V with respect to Ψ. Clearly every subspace of a totally singular subspace is also totally singular and in the case of linear groups all subspaces are totally singular. If Ψ is a non-singular form then by the Witt theorem all maximal totally singular subspaces have the same dimension known as the *Witt index* of Ψ.

The geometry $\mathcal{G} = \mathcal{G}(G)$ of a classical group G is the set of all proper totally singular subspaces in the natural module V with respect to the invariant form Ψ together with a symmetrical binary incidence relation $*$ under which two subspaces are incident if and only if one of them contains the other one. In the case of a linear group we obtain the projective geometry associated with the natural module and in the remaining cases we obtain various polar spaces.

By the definition every element of a classical group geometry is incident to itself which means that the relation $*$ is reflexive. One can consider $\mathcal{G}$ as a graph on the set of elements whose edges are pairs of incident elements. Since two subspaces of the same dimension are incident if and only if they coincide, one can see (ignoring the loops) that the graph is multipartite. Two vertices are contained in the same part if and only if they have the same dimension as subspaces of V. It is natural to define the type of an element to be the dimension of the corresponding subspace. The Witt theorem and its trivial analogue for the case of linear groups imply that every maximal set of pairwise incident elements of $\mathcal{G}$ (a maximal clique in graph-theoretical terms) contains exactly one element of each type. This construction suggests the definition of geometry as introduced by J. Tits in the 1950s.

Geometries form a special class of *incidence systems*. An incidence system is a quadruple $(\mathcal{G}, *, t, I)$ where $\mathcal{G}$ is the set of elements, $*$ is a binary reflexive symmetric incidence relation on $\mathcal{G}$ and t is a type function which prescribes for every element from $\mathcal{G}$ its type which is an element from the set I of possible types; two different elements of the same type are never incident. We will usually refer to an incidence system $(\mathcal{G}, *, t, I)$ simply by writing $\mathcal{G}$, assuming that $*$, t and I are clear from the context. The number of types in an incidence system (that is the size of I) is called the *rank*. Unless stated otherwise, we will always assume that $I = \{1, 2, ..., n\}$ for an incidence system of rank n and write $\mathcal{G}^i$ for the set of elements of type i in $\mathcal{G}$, that is for $t^{-1}(i)$.

An incidence system $\mathcal{G}$ of rank n can be considered (ignoring loops) as an n-partite graph with parts $\mathcal{G}^1, ..., \mathcal{G}^n$. An incidence system is *connected* if it is connected as a graph.

A set Φ of pairwise incident elements in an incidence system is called a *flag*. In this case $|\Phi|$ and $t(\Phi)$ are the *rank* and the *type* of Φ, respectively. If $\mathcal{G}$ is an incidence system of rank n over the set I of types then $n - |\Phi|$ and $I \setminus t(\Phi)$ are the *corank* and the *cotype* of Φ, respectively. Let Φ be a flag in an incidence system $\mathcal{G}$. The *residual incidence system* $\mathrm{res}_{\mathcal{G}}(\Phi)$ of Φ in $\mathcal{G}$ (or simply *residue*) is the quadruple $(\mathcal{G}_\Phi, *_\Phi, t_\Phi, I_\Phi)$ where

$$\mathcal{G}_\Phi = \{x \mid x \in \mathcal{G},\ x * y \text{ for every } y \in \Phi\} \setminus \Phi,$$

$I_\Phi = I \setminus t(\Phi)$, $*_\Phi$ is the restriction of $*$ to $\mathcal{G}_\Phi$ and t_Φ is the restriction of t to $\mathcal{G}_\Phi$ The notion of residue corresponds to that of link, more common in topology. For a flag consisting of a single element x its residue will be denoted by $\text{res}_{\mathscr{G}}(x)$ rather than by $\text{res}_{\mathscr{G}}(\{x\})$. It is easy to see that one can construct an arbitrary residue inductively, producing at each step the residue of a single element.

Definition 1.1.1 *A geometry is an incidence system* $(\mathscr{G}, *, t, I)$ *for which the following two conditions hold:*

(i) *every maximal flag contains exactly one element of each type;*

(ii) *for every* $i, j \in t(\mathscr{G})$ *the graph on* $\mathscr{G}^i \cup \mathscr{G}^j$ *in which two elements are adjacent if they are incident in* $\mathscr{G}$ *is connected, and a similar condition holds for every residue in* $\mathscr{G}$ *of rank at least* 2.

The graph on the set of elements of a geometry $\mathscr{G}$ in which two distinct elements are adjacent if they are incident in $\mathscr{G}$ is called the *incidence graph* of $\mathscr{G}$. The incidence graphs of geometries of rank n are characterized as n-partite graphs with the following properties: (i) every maximal clique contains exactly one vertex from each part; (ii) the subgraph induced by any two parts is connected and a similar connectivity condition holds for every residue of rank at least 2. It is easy to see that a residue of a geometry is again a geometry.

Let $(\mathscr{G}_1, *_1, t_1, I_1)$ and $(\mathscr{G}_2, *_2, t_2, I_2)$ be two geometries whose sets of elements and types are disjoint. The *direct sum* of $\mathscr{G}_1$ and $\mathscr{G}_2$ is a geometry whose element set is $\mathscr{G}_1 \cup \mathscr{G}_2$, whose set of types is $I_1 \cup I_2$, whose incidence relation and type function coincide respectively with $*_i$ and t_i when restricted to $\mathscr{G}_i$ for $i = 1$ and 2 and where every element from $\mathscr{G}_1$ is incident to every element from $\mathscr{G}_2$.

The above definitions of residue and direct sum have the following motivation in the context of geometries of classical groups. Let G be a classical group with a natural module V and the invariant form Ψ. Let $\mathscr{G} = \mathscr{G}(G)$ be the geometry of G as defined above. Let W be an element of $\mathscr{G}$ that is a totally singular subspace of V with respect to Ψ. It is easy to see that $\text{res}_{\mathscr{G}}(W)$ is the direct sum of two geometries $\text{res}^-_{\mathscr{G}}(W)$ and $\text{res}^+_{\mathscr{G}}(W)$, where the former is the projective geometry of all proper subspaces of W and the latter is formed by the totally singular subspaces containing W and can be described as follows. Let

$$W^\perp = \{v \mid v \in V, \Psi(v, w) = 0 \;\text{ for every }\; w \in W\}$$

be the orthogonal complement of W. Then $W \leq W^\perp$ and Ψ induces on $U = W^\perp/W$ a non-singular form Ψ'. The elements of $\text{res}^+_{\mathscr{G}}(W)$ are the subspaces of U totally singular with respect to Ψ' with the incidence relation given by inclusion. So $\mathrm{res}^{+}_{\mathscr{G}}(W)$ is the geometry of the classical group having U as natural module and Ψ' as invariant form. Certainly $\mathrm{res}^{-}_{\mathscr{G}}(W)$ or $\mathrm{res}^{+}_{\mathscr{G}}(W)$ or both can be empty and one can easily figure out when this happens. In any case the observation is that the class of direct sums of geometries of classical groups is closed under taking residues.

By introducing geometries of classical groups we started considering the totally isotropic subspaces of their natural modules as abstract elements preserving from their origin in the vector space the incidence relation and type function. It turns out that in most cases the vector space can be uniquely reconstructed from the geometry and moreover the geometry itself to a certain extent is characterized by its local properties, namely by the structure of residues. The theory and classification of geometries can be developed quite deeply without making any assumption on their automorphism groups. But our primary interest is in so-called flag-transitive geometries to be introduced in the next section.

1.2 Morphisms of geometries

Let $\mathscr{H}$ and $\mathscr{G}$ be geometries (or more generally incidence systems). A *morphism* of geometries is a mapping $\varphi : \mathscr{H} \to \mathscr{G}$ of the element set of $\mathscr{H}$ into the element set of $\mathscr{G}$ which maps incident pairs of elements onto incident pairs and preserves the type function. A bijective morphism is called an *isomorphism*.

A surjective morphism $\varphi : \mathscr{H} \to \mathscr{G}$ is said to be a *covering* of $\mathscr{G}$ if for every non-empty flag Φ of $\mathscr{H}$ the restriction of φ to the residue $\mathrm{res}_{\mathscr{H}}(\Phi)$ is an isomorphism onto $\mathrm{res}_{\mathscr{G}}(\varphi(\Phi))$. In this case $\mathscr{H}$ is a *cover* of $\mathscr{G}$ and $\mathscr{G}$ is a *quotient* of $\mathscr{H}$. If every covering of $\mathscr{G}$ is an isomorphism then $\mathscr{G}$ is said to be *simply connected*. Clearly a morphism is a covering if its restriction to the residue of every element (considered as a flag of rank 1) is an isomorphism. If $\psi : \widetilde{\mathscr{G}} \to \mathscr{G}$ is a covering and $\widetilde{\mathscr{G}}$ is simply connected, then ψ is the *universal covering* and $\widetilde{\mathscr{G}}$ is the *universal cover* of $\mathscr{G}$. The universal cover of a geometry exists and it is uniquely determined up to isomorphism. If $\varphi : \mathscr{H} \to \mathscr{G}$ is any covering then there exists a covering $\chi : \widetilde{\mathscr{G}} \to \mathscr{H}$ such that ψ is the composition of χ and φ.

A morphism $\varphi : \mathscr{H} \to \mathscr{G}$ of arbitrary incidence systems is called an *s-covering* if it is an isomorphism when restricted to every residue of rank at least s. This means that if Φ is a flag whose cotype is less than or equal to s, then the restriction of φ to $\mathrm{res}_{\mathscr{H}}(\Phi)$ is an isomorphism. An incidence system, every s-cover of which is an isomorphism, is said to be *s-simply connected*. The universal s-cover of a geometry exists in the class of incidence systems and it might or might not be a geometry. In the present work we will mainly use the notion of s-covers either to deal with concrete morphisms of geometries or to establish s-simple connectedness. For these purposes we can stay within the class of geometries. It must be clear that in the case $s = n-1$ "s-covering" and "covering" mean the same thing.

An isomorphism of a geometry onto itself is called an *automorphism*. By the definition an isomorphism preserves the types. Sometimes we will need a more general type of automorphisms which permute types. We will refer to them as *diagram automorphisms*.

The set of all automorphisms of a geometry $\mathcal{G}$ obviously forms a group called the *automorphism group* of $\mathcal{G}$ and denoted by $\mathrm{Aut}\,\mathcal{G}$. An automorphism group G of $\mathcal{G}$ (that is a subgroup of $\mathrm{Aut}\,\mathcal{G}$) is said to be *flag-transitive* if any two flags Φ_1 and Φ_2 in $\mathcal{G}$ of the same type (that is with $t(\Phi_1) = t(\Phi_2)$) are in the same G-orbit. Clearly an automorphism group is flag-transitive if and only if it acts transitively on the set of maximal flags in $\mathcal{G}$. A geometry $\mathcal{G}$ possessing a flag-transitive automorphism group is called *flag-transitive*.

A flag-transitive geometry can be described in terms of certain subgroups and their cosets in a flag-transitive automorphism group in the following way. Let $\mathcal{G}$ be a geometry of rank n and G be a flag-transitive automorphism group of $\mathcal{G}$. Let $\Phi = \{x_1, x_2, ..., x_n\}$ be a maximal flag in $\mathcal{G}$ where x_i is of type i. Let $G_i = G(x_i)$ be the stabilizer of x_i in G. The subgroups G_1, G_2, ..., G_n are called the *maximal parabolic subgroups* or just *maximal parabolics* associated with the action of G on $\mathcal{G}$. When talking about n maximal parabolic subgroups associated with an action on a rank n geometry we will always assume that the elements which they stabilize form a maximal flag. By the flag-transitivity assumption G acts transitively on the set $\mathcal{G}^i$ of elements of type i in $\mathcal{G}$. So there is a canonical way to identify $\mathcal{G}^i$ with the set of right cosets of G_i in G by associating with $y \in \mathcal{G}^i$ the coset G_ih such that $x_i^h = y$. This coset consists of all the elements of G which map x_i onto y (assuming that action is on the right). Now with y as above let z be an element of type j which corresponds to the coset G_jk. By the flag-transitivity assumption y and z are incident if and only if there is an element g in G which maps the pair (x_i, x_j) onto the pair (y, z). It is obvious that g must be in the intersection $G_ih \cap G_jk$ and each element from the intersection can be taken for g. Thus y and z are incident if and only if the cosets G_ih and G_jk have a non-empty intersection. Notice that if the intersection is non-empty, it is a right coset of $G_i \cap G_j$. In this way we arrive at the following.

Proposition 1.2.1 *Let $\mathcal{G}$ be a geometry of rank n over the set $I = \{1, 2, ..., n\}$ of types and G be a flag-transitive automorphism group of $\mathcal{G}$. Let $\Phi = \{x_1, x_2, ..., x_n\}$ be a maximal flag in $\mathcal{G}$ and $G_i = G(x_i)$ be the stabilizer of x_i in G. Let $\mathcal{G}(G)$ be the incidence system whose elements of type i are the right cosets of G_i in G and in which two elements are incident if and only if the intersection of the corresponding cosets is non-empty. Then $\mathcal{G}(G)$ is a geometry and the mapping*

$$\eta : y \mapsto G_i h$$

(where $y \in \mathcal{G}^i$ and $x_i^h = y$) establishes an isomorphism of $\mathcal{G}$ onto $\mathcal{G}(G)$. □

1.3 Amalgams

Discussions in the previous section and particularly (1.2.1) lead to the following.

Definition 1.3.1 *A (finite)* amalgam $\mathscr{A}$ *of rank n is a finite set H such that for every $1 \le i \le n$ there are a subset H_i in H and a binary operation $*_i$ defined on H_i such that the following conditions hold:*

(i) *$(H_i, *_i)$ is a group for $1 \le i \le n$;*
(ii) *$H = \bigcup_{i=1}^n H_i$;*
(iii) *$\bigcap_{i=1}^n H_i \neq \emptyset$;*
(iv) *if $x, y \in H_i \cap H_j$ for $1 \le i < j \le n$ then $x *_i y = x *_j y$.*

We will usually write $\mathscr{A} = \{H_i \mid 1 \le i \le n\}$ for the amalgam $\mathscr{A}$ as in the above definition. Whenever x and y are in the same H_i their product $x *_i y$ is defined and it is independent of the choice of i. We will normally denote this product simply by xy. Since $B := \bigcap_{i=1}^n H_i$ is non-empty, one can easily see that B contains the identity element of $(H_i, *_i)$ for every $1 \le i \le n$. Moreover, all these identity elements must be equal. The reader may notice that a more common definition of amalgams in terms of morphisms is essentially equivalent to the above one.

If $(G, *)$ is a group, $H_1, ..., H_n$ are subgroups of G and $*_1, ..., *_n$ are the restrictions of $*$ to these subgroups, then $\mathscr{A} = \{H_i \mid 1 \le i \le n\}$ is an amalgam. This is the most important example of an amalgam, but at the same time it is not very difficult to construct an example of an amalgam which is not isomorphic to a family of subgroups of a group. The amalgam $\mathscr{A}$ as above is said to be isomorphic to an amalgam $\mathscr{A}' = \{H_i' \mid 1 \le i \le n\}$ if there is a bijection of H onto H' which induces an isomorphism of $(H_i, *_i)$ onto $(H_i', *_i')$ for every $1 \le i \le n$.

Definition 1.3.2 *A group G is said to be a* completion *of an amalgam $\mathscr{A} = \{H_i \mid 1 \le i \le n\}$ if there is a mapping φ of H into G such that*

(i) *G is generated by the image of φ,*
(ii) *for every $1 \le i \le n$ the restriction of φ to H_i is a group homomorphism with respect to $*_i$ and the group operation in G.*

If φ is injective then the completion G is said to be faithful.

Thus an amalgam $\mathscr{A}$ is isomorphic to a family of subgroups of a group if and only if $\mathscr{A}$ possesses a faithful completion. If G is a faithful completion of $\mathscr{A}$ then we will usually identify $\mathscr{A}$ and its image in G.

There is a completion $U(\mathscr{A})$ of $\mathscr{A}$ known as the *universal completion*, of which any completion is a homomorphic image. The group $U(\mathscr{A})$ has the following definition in terms of generators and relations: the generators are all the elements of H; the relations are all the equalities of the form $xyz^{-1} = 1$ where x and y are (possibly equal) elements contained in H_i for some i and $z = x *_i y$. It is easy to see that $U(\mathscr{A})$ is a completion of $\mathscr{A}$ with respect to the mapping ψ which sends every $x \in H$ onto the corresponding generator of $U(\mathscr{A})$. Moreover, if G is an arbitrary completion of $\mathscr{A}$ with respect to a mapping φ then there is a unique homomorphism $\chi : U(\mathscr{A}) \to G$ such that φ is the composition of ψ and χ. Finally, $\mathscr{A}$ possesses a faithful completion if and only if $U(\mathscr{A})$ is a faithful completion.

Let G, $\mathscr{G}$ and the G_i be as in (1.2.1). The amalgam $\mathscr{A} = \{G_i \mid 1 \le i \le n\}$ is called *the amalgam of maximal parabolic subgroups* in G associated with the flag Φ. The geometry $\mathscr{G}(G)$ should be denoted by $\mathscr{G}(G, \mathscr{A})$ since its structure is determined not only by G by also by the amalgam $\mathscr{A}$ and by the embedding of $\mathscr{A}$ in G. We can reformulate (1.2.1) as follows.

Proposition 1.3.3 *Let G be a flag-transitive automorphism group of a geometry $\mathscr{G}$ of rank n and $\mathscr{A} = \{G_i \mid 1 \le i \le n\}$ be the amalgam of maximal parabolic subgroups associated with a maximal flag. Let $\mathscr{G}(G, \mathscr{A})$ be the incidence system whose elements of type i are the right cosets of G_i in G and in which two elements are incident if and only if the intersection of the corresponding cosets is non-empty. Then $\mathscr{G}$ and $\mathscr{G}(G, \mathscr{A})$ are isomorphic.* □

Notice that by the above proposition the residues of $\mathscr{G}$ are uniquely determined by the amalgam $\mathscr{A}$. That is, $\mathrm{res}_{\mathscr{G}}(x_i)$ is isomorphic to $\mathscr{G}(G_i, \mathscr{A}_i)$ where $\mathscr{A}_i = \{G_i \cap G_j \mid 1 \le j \le n,\ j \ne i\}$.

For a subset $J \subseteq I = \{1, 2, ..., n\}$ let $G_J = \bigcap_{i \in J} G_i$ be the elementwise stabilizer in G of the flag $\{x_i \mid i \in J\}$. The subgroup G_J is a *parabolic* *subgroup* of rank r where $r = |I| - |J|$. For $i, j \in I$ we write G_{ij} instead of $G_{\{i,j\}}$. The parabolic subgroups of rank $n-1$ are the maximal parabolics. The parabolic subgroups of rank 1 are known as *minimal parabolics* and the subgroup $B = G_I$ is called the *Borel subgroup*. We will usually write P_i to denote the minimal parabolic $G_{I \setminus \{i\}}$ and P_{ij} to denote the rank 2 parabolic $G_{I \setminus \{i,j\}}$.

1.4 Geometrical amalgams

In view of (1.3.3) the following question naturally arises.

Q. Let G be a group, $G_1, G_2, ..., G_n$ be subgroups of G and $\mathscr{A} = \{G_i \mid 1 \le i \le n\}$ be the amalgam formed by these subgroups. Under what circumstances is the incidence system $\mathscr{G} = \mathscr{G}(G, \mathscr{A})$ a geometry and the natural action of G on $\mathscr{G}$ flag-transitive?

Below we discuss the answer to this question as given in [Ti74].

The set $\Phi = \{G_1, G_2, ..., G_n\}$ is a flag in $\mathscr{G}$ since each G_i contains the identity element and Φ is a maximal flag since for $1 \le i \le n$ and $g \in G$ either $G_i g = G_i$ or $G_i g \cap G_i = \emptyset$. A set $\Psi = \{G_{i_1} h_1, ..., G_{i_m} h_m\}$ is a flag in $\mathscr{G}$ if and only if $G_{i_j} h_j \cap G_{i_k} h_k \neq \emptyset$ for all j, k with $1 \le j, k \le m$ (which implies particularly that $i_j \neq i_k$). We say that the flag Ψ is *standard* if the intersection $\bigcap_{j=1}^m G_{i_j} h_j$ is non-empty and contains an element h, say. In this case $\Psi = \{G_{i_1}, ..., G_{i_m}\}^h$, which means that Ψ is the image under h of a subflag in Φ. This shows that every standard flag is contained in a standard maximal flag and G acts transitively on the set of standard flags of each type. Clearly G cannot map a standard flag onto a non-standard one. Thus the necessary and sufficient condition for flag-transitivity of the natural action of G on $\mathscr{G}$ is absence of non-standard flags.

The proof of the following result uses elementary group theory only (compare Sections 10.1.3 and 10.1.4 in [Pasi94]).

Lemma 1.4.1 *The incidence system $\mathscr{G}(G, \mathscr{A})$ does not contain non-standard flags if and only if the following equivalent conditions hold:*

(i) *if J, K, L are subsets of I and g, h, f are elements in G such that the cosets $G_J g$, $G_K h$, $G_L f$ have pairwise non-empty intersection, then $G_J g \cap G_K h \cap G_L f \neq \emptyset$;*

(ii) *for $i, j \in I$ and $J \subseteq I \setminus \{i, j\}$ if $g \in G_J$ and $G_i \cap G_j g \neq \emptyset$ then $G_J \cap G_i \cap G_j g \neq \emptyset$.* □

One may notice that, in general, existence of non-standard flags in $\mathscr{G}(G, \mathscr{A})$ depends not only on the structure of $\mathscr{A}$ but also on the structure of G.

The connectivity condition in (1.1.1 (ii)) is also easy to express in terms of parabolic subgroups. By the standard principle the graph on $\mathscr{G}^i \cup \mathscr{G}^j$ is connected if and only if G is generated by the subgroups G_i and G_j. This gives the following.

Lemma 1.4.2 *The incidence system $\mathscr{G}(G, \mathscr{A})$ satisfies the condition (ii) in (1.1.1) if and only if for every 2-element subset $\{i, j\} \subseteq I$ the subgroups G_i and G_j generate G.* □

Finally let K be the kernel of the action of G on $\mathscr{G}(G, \mathscr{A})$. It is straightforward that K is the largest subgroup in the Borel subgroup $B = \bigcap_{i=1}^{n} G_i$, which is normal in G_i for all i with $1 \leq i \leq n$ (equivalently, normal in G). In particular the action of G on $\mathscr{G}(G, \mathscr{A})$ is faithful if and only if the Borel subgroup contains no non-identity subgroup normal in G.

1.5 Universal completions and covers

The fact that the structure of residues in $\mathscr{G}(G, \mathscr{A})$ is determined solely by $\mathscr{A}$ plays a crucial rôle in the description of the coverings of $\mathscr{G}(G, \mathscr{A})$.

Let $\mathscr{G}$ be a geometry, G be a flag-transitive automorphism group of $\mathscr{G}$ and $\mathscr{A} = \{G_i \mid 1 \leq i \leq n\}$ be the amalgam of maximal parabolic subgroups associated with the action of G on $\mathscr{G}$. Then on the one hand $\mathscr{G} \cong \mathscr{G}(G, \mathscr{A})$ and on the other hand G is a faithful completion of $\mathscr{A}$. Let G' be another faithful completion of $\mathscr{A}$ and let

$$\varphi : G' \to G$$

be an $\mathscr{A}$-homomorphism, *i.e.* a homomorphism of G' onto G whose restriction to $\mathscr{A}$ is the identity mapping. As usual we identify $\mathscr{A}$ with its images in G' and G. The following result is straightforward.

Lemma 1.5.1 *In the above terms the mapping of $\mathscr{G}(G', \mathscr{A})$ onto $\mathscr{G}(G, \mathscr{A})$ induced by φ is a covering of geometries.* □

In the above construction we could take G' to be the universal completion $U(\mathscr{A})$ of $\mathscr{A}$. The following result of fundamental importance was proved independently in [Pasi85], [Ti86] and an unpublished manuscript by S.V. Shpectorov.