ONE

DERIVATION OF THE COMPRESSIBLE NAVIER-STOKES EQUATIONS

1–1 Introduction

The Navier-Stokes equations, together with the energy and continuity equations, are the cornerstones of continuum fluid dynamics. The equations were first derived for incompressible flow by Navier in 1822 and subsequently placed on a more rigorous basis by St. Venant, Poisson and Stokes.

Although the final form of the equations does not seem to be in dispute, the route to their formulation is a matter of some difference in opinion. It is not evident that all procedures are mutually consistent.

The stimulus for formulation of the laws of friction in a fluid was the success of the equations of elasticity, i.e., the generalized Hooke's law, which postulates that the components of the stress and strain tensors are linearly related by a number of constants whose values depend on the material under consideration. If, however, the material is *homogeneous* and *isotropic*, and if it is also stipulated that the equations be *invariant* under a rotation of coordinates, i.e., their form be the same for an arbitrarily-oriented set of orthogonal coordinates, the number of independent material constants reduces to two. In the analogous fluid dynamic situation, it is assumed that the component stresses are linearly dependent on the *rate-of-strain* rather than the strain. This assumption is fundamental to the derivation of the Navier-Stokes equations.

Three independent postulates furnish the fundamental relations of fluid dynamics. They are conservation of mass, Newton's second law of motion and conservation of energy. The fundamental relations of fluid dynamics are stated in terms of particle properties, e.g., the principle of conservation of mass states that *particles* of matter can neither be created nor destroyed. In order to apply these relations to a fluid we introduce the concept of a field. A field is a region whose properties are defined as functions of space and time. A one-to-one correspondence between particle properties and field properties in a fluid is achieved by assuming that as particles move through the field they take on the properties of the field point that they occupy. Using this concept we can apply the fundamental relations to a continuous fluid, i.e., a fluid not composed of discrete particles.

We first consider the principle of conservation of mass. Without loss of generality we consider a cubical control volume as shown in Figure 1–1. We assume that the properties of the flow are known at the center of the element. Some of these properties are the density ρ, the pressure P, and the velocity components u, v, w in the x, y, z directions, respectively. Further, we assume that the values of these quantities on the faces of the element are adequately represented by a first-order Taylor series expansion about the center of the element. This assumption fundamentally depends on the fluid being continuous and homogeneous, i.e., the composition is the same throughout.

Considering the mass flow into and out of the control volume, and neglecting the possibility of the production of matter inside the control volume, i.e., sources or sinks of matter, the principle of conservation of mass stated as a rate equation is

$$\begin{pmatrix}\text{The rate at which}\\ \text{mass leaves the}\\ \text{control volume}\\ \text{through the faces}\end{pmatrix} - \begin{pmatrix}\text{The rate at which}\\ \text{mass enters the}\\ \text{control volume}\\ \text{through the faces}\end{pmatrix} = \begin{pmatrix}\text{The rate at which}\\ \text{mass decreases}\\ \text{within the}\\ \text{control volume}\end{pmatrix}$$

If, for convenience, we consider that a fluid flows diagonally through the field represented by the elemental volume from the lower rear left corner to the upper front right corner, we see that mass leaves the volume through the right hand,

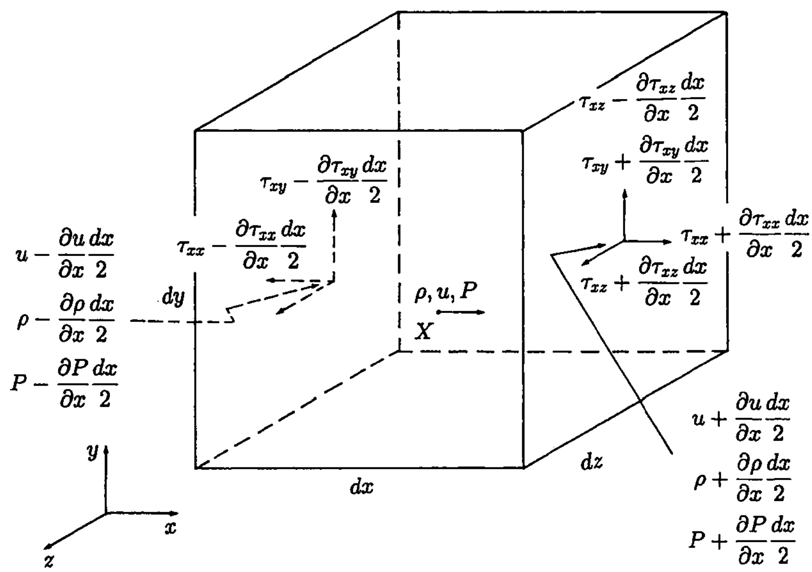

Figure 1–1. Elemental control volume.

front and upper faces and enters the volume through the left hand, rear and lower faces. The rate at which mass leaves the control volume is then

$$\begin{pmatrix}\text{The rate at which}\\ \text{mass leaves the}\\ \text{control volume}\\ \text{through the faces}\end{pmatrix} = \left(\rho + \frac{\partial \rho}{\partial x}\frac{dx}{2}\right)\left(u + \frac{\partial u}{\partial x}\frac{dx}{2}\right)dydz$$
$$+ \left(\rho + \frac{\partial \rho}{\partial y}\frac{dy}{2}\right)\left(v + \frac{\partial v}{\partial y}\frac{dy}{2}\right)dxdz$$
$$+ \left(\rho + \frac{\partial \rho}{\partial z}\frac{dz}{2}\right)\left(w + \frac{\partial w}{\partial z}\frac{dz}{2}\right)dxdy$$

Similarly, the rate at which mass enters the control volume is

$$\begin{pmatrix}\text{The rate at which}\\ \text{mass enters the}\\ \text{control volume}\\ \text{through the faces}\end{pmatrix} = \left(\rho - \frac{\partial \rho}{\partial x}\frac{dx}{2}\right)\left(u - \frac{\partial u}{\partial x}\frac{dx}{2}\right)dydz$$
$$+ \left(\rho - \frac{\partial \rho}{\partial y}\frac{dy}{2}\right)\left(v - \frac{\partial v}{\partial y}\frac{dy}{2}\right)dxdz$$
$$+ \left(\rho - \frac{\partial \rho}{\partial z}\frac{dz}{2}\right)\left(w - \frac{\partial w}{\partial z}\frac{dz}{2}\right)dxdy$$

Without loss of generality, we assume that the rate at which mass leaves the control volume is larger than the rate at which mass enters the control volume. The rate at which mass decreases within the control volume is then

$$\begin{pmatrix}\text{The rate at which}\\ \text{mass decreases}\\ \text{within the}\\ \text{control volume}\end{pmatrix} = -\frac{\partial \rho}{\partial t}\,dxdydz$$

Combining these results and neglecting the higher-order terms, i.e., terms of order $(dx)^2$, $(dy)^2$, $(dz)^2$, yields the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \qquad (1-1)$$

Here note that no restrictions are placed on the observer's frame of reference and that the only assumptions with respect to the fluid are that it be continuous and homogeneous.

From Newton's second law, which is restricted to an inertial frame of reference, we have

$$m\,\frac{D\vec{V}}{Dt} = \sum \vec{F}_{\text{ext}} \qquad (1-2)$$

Here, m is the mass of a particle and $D\vec{V}/Dt$ is the total acceleration of the particle. Again we introduce the field concept in order to apply Newton's second law to a continuous fluid. The total acceleration of a fluid particle moving through the velocity field is composed of a convective contribution due to movement of the particle through the velocity field, and a local contribution due to the change in the velocity field with time at any position within the field, i.e., unsteady field effects.

Let us now determine the x component of the convective acceleration of a particle moving through the velocity field. If the x component of the velocity of the particle at some point in the field is u_1, then the x component of its velocity at some other neighboring point is u_2 and can be determined using a first-order Taylor series expansion. Thus

$$u_2 = u_1 + \left.\frac{\partial u}{\partial x}\right|_1 dx + \left.\frac{\partial u}{\partial y}\right|_1 dy + \left.\frac{\partial u}{\partial z}\right|_1 dz + \text{h.o.t.}^\dagger \tag{1-3}$$

Due to the particle's movement through the field its velocity is changing. Thus, the particle is accelerating. Hence we have

$$\begin{pmatrix}\text{The } x \text{ component}\\ \text{of the convective}\\ \text{acceleration}\end{pmatrix} = \lim_{(t_2-t_1)\to 0} \frac{u_2 - u_1}{t_2 - t_1} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

since $u = dx/dt$, $v = dy/dt$ and $w = dz/dt$.

The x component of the local acceleration is due to unsteady field effects, i.e., the field velocity at a fixed point in the field is a function of time. Hence, the x component of the local acceleration is $\partial u/\partial t$. The total or substantial acceleration is the sum of the local and convective accelerations. Thus, the x component of the total acceleration is

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \tag{1-4a}$$

Similarly, the y and z components of the total acceleration are, respectively

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \tag{1-4b}$$

and

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \tag{1-4c}$$

Here D/Dt is the substantial or Eulerian derivative. In Cartesian coordinates

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\left(\frac{\partial}{\partial x}\right) + v\left(\frac{\partial}{\partial y}\right) + w\left(\frac{\partial}{\partial z}\right)$$

† h.o.t. means higher-order terms.

The external forces acting on the control volume $\vec{F}_{\text{ext}}$ are divided into body or volume forces and into surface forces. A body force is one which acts on the entire mass of the element. Examples are gravity and electromagnetic forces. We use X, Y, Z to represent the x, y and z components of the body force per unit mass (see Figure 1–1). Surface forces act only at the surface of the control volume. Examples are viscous shearing forces and normal pressure forces. Figure 1–1 shows the stresses (force/unit area) acting on the elemental volume. Here, the notation is such that the first subscript indicates the face on which the stress is acting and the second the direction in which it is acting. Thus, for example, τ_{xy} is the stress acting on the face perpendicular to the x-axis in the y direction. The components of the stress on the faces of the elemental control volume form a shearing stress tensor, which describes the state of stress in the fluid. The matrix of the shearing stress tensor is

$$[\tau] = \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix} \tag{1-5}$$

From Figure 1–1 we see that the acceleration in the x direction is equal to the sum of the body force term, a normal force term involving τ_{xx} (which includes the pressure), and two shearing force terms involving τ_{yx} and τ_{zx}. Thus, the x component of Newton's second law is

$$\frac{Du}{Dt} = X + \frac{1}{\rho}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) \tag{1-6a}$$

Similarly, the equations in the y and z directions are

$$\frac{Dv}{Dt} = Y + \frac{1}{\rho}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) \tag{1-6b}$$

$$\frac{Dw}{Dt} = Z + \frac{1}{\rho}\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) \tag{1-6c}$$

Equations (1–6) are known as Cauchy's equations of motion.

The shearing stress tensor is symmetrical, i.e., $\tau_{yx} = \tau_{xy}$, $\tau_{xz} = \tau_{zx}$ and $\tau_{yz} = \tau_{zy}$. In order to show this, consider one of the faces of the elemental volume as illustrated in Figure 1–2. Conservation of angular momentum requires that the torque acting on the element be equal to the time rate-of-change of the angular momentum on the element. Thus

$$\tau_{yx}\frac{h}{2}h - \tau_{xy}\frac{h}{2}h = \dot{\omega} I_{zz} \tag{1-7}$$

where I_{zz} is the polar moment of inertia and $\dot{\omega}$ is the angular acceleration. For a square of unit depth we have

$$\tau_{yx}\frac{h^2}{2} - \tau_{xy}\frac{h^2}{2} = \dot{\omega}\frac{h^4}{6}$$

or

$$\tau_{yx} - \tau_{xy} = \dot{\omega}\frac{h^2}{3}$$

The angular velocity due to the shearing stresses is

$$\omega = 3\int \frac{(\tau_{yx} - \tau_{xy})}{h^2}\,dt \tag{1-8}$$

As the volume (in this two-dimensional case the area) of the element approaches zero, i.e., $h \to 0$, the angular velocity must approach zero. This is only possible if the integrand in Eq. (1–8) is zero. Hence,

$$\tau_{yx} = \tau_{xy} \tag{1-9}$$

The normal stress τ_{xx} includes the pressure which is assumed to be a scalar and hence independent of direction. Therefore, for convenience we separate out the pressure by writing

$$\tau_{xx} = \tau'_{xx} - P \tag{1-10}$$

where the minus sign indicates that the pressure exerts a force acting on the elemental area in a direction opposite to the positive outer normal. Equation (1–6a) thus becomes

$$\frac{Du}{Dt} = X - \frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{1}{\rho}\left(\frac{\partial \tau'_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) \tag{1-11a}$$

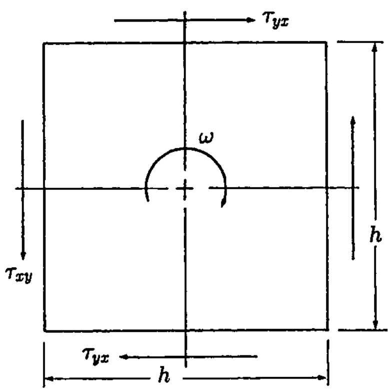

Figure 1–2. Two-dimensional angular momentum.

and similarly Eqs. (1–6b) and (1–6c) become

$$\frac{Dv}{Dt} = Y - \frac{1}{\rho}\frac{\partial P}{\partial y} + \frac{1}{\rho}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau'_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) \tag{1-11b}$$

$$\frac{Dw}{Dt} = Z - \frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{1}{\rho}\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau'_{zz}}{\partial z}\right) \tag{1-11c}$$

The matrix of the stress tensor becomes

$$[\tau'] = \begin{bmatrix} \tau'_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau'_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau'_{zz} \end{bmatrix} \tag{1-12}$$

1–2 Stress–Rate-of-Strain Relation

We are now at a critical point in the discussion: the form of the stress–rate-of-strain relation for a viscous flow. The most general assumption for the stress–rate-of-strain relationship, which is still linear, is that an arbitrary stress component is a linear function of all the rates-of-strain; e.g.

$$\tau'_{xx} = \alpha_1\dot\epsilon_{xx} + \alpha_2\dot\epsilon_{xy} + \alpha_3\dot\epsilon_{xz} + \cdots \alpha_9\dot\epsilon_{zz} \tag{1-13}$$

where the α_n's are independent material properties, at most functions of the thermodynamic state variables, and independent of the rates-of-strain, i.e., the ϵ's. Here, the same subscripting convention used for the stresses is used for the rates-of-strain. As there are nine stresses and nine rates-of-strain, a maximum of 81 independent material properties is possible.

It has already been established that the stress tensor is symmetric. In addition, it is subsequently shown that the rate-of-strain tensor is also symmetric. Further, if it is assumed that the fluid is isotropic and homogeneous, together with the stipulation that the stress–rate-of-strain relation be invariant under a rotation of orthogonal coordinates, the number of independent properties reduces to two. The resulting equations then take the form (see Liepmann and Roshko [Liep57])

$$\tau'_{xx} = 2\mu\dot\epsilon_{xx} + \lambda(\dot\epsilon_{xx} + \dot\epsilon_{yy} + \dot\epsilon_{zz}) \tag{1-14a}$$

$$\tau'_{yy} = 2\mu\dot\epsilon_{yy} + \lambda(\dot\epsilon_{xx} + \dot\epsilon_{yy} + \dot\epsilon_{zz}) \tag{1-14b}$$

$$\tau'_{zz} = 2\mu\dot\epsilon_{zz} + \lambda(\dot\epsilon_{xx} + \dot\epsilon_{yy} + \dot\epsilon_{zz}) \tag{1-14c}$$

and

$$\tau_{xy} = \tau_{yx} = 2\mu\dot\epsilon_{yx} \tag{1-15a}$$

$$\tau_{yz} = \tau_{zy} = 2\mu\dot\epsilon_{zy} \tag{1-15b}$$

$$\tau_{xz} = \tau_{zx} = 2\mu\dot\epsilon_{xz} \tag{1-15c}$$

Here μ is the well known dynamic viscosity, and λ is the second coefficient of viscosity, or the dilatational viscosity.

To obtain the components of the rate-of-strain tensor and hence the components of the stress tensor in terms of more convenient and physically meaningful quantities, we consider the angular velocity and deformation of a fluid element.

1–3 Angular Velocity and Fluid Deformation

A continuous fluid is assumed to be composed of arbitrarily small particles (mathematical points). Since these points are infinitely small it is not reasonable to consider the rotation of a single particle. However, by considering the behavior of the intersection of two originally orthogonal fluid lines, i.e., two mathematical lines always connecting the same particles—two fluid curves, during a short interval of time the local rotation and distortion of the fluid is determined.

Consider two lines lying in the xy plane (see Figure 1–3). The z component of angular velocity is defined as the average rotation of the tangents to these fluid lines at their intersection divided by the time interval. Taking the limit as $\Delta t \to 0$

$$\omega_z = \lim_{\Delta t \to 0} \frac{1}{2}\left(\frac{\Delta\alpha + \Delta\beta}{\Delta t}\right) = \frac{1}{2}\left(\frac{d\alpha}{dt} + \frac{d\beta}{dt}\right) \qquad (1-16)$$

We now wish to determine the angular velocity in terms of the translational

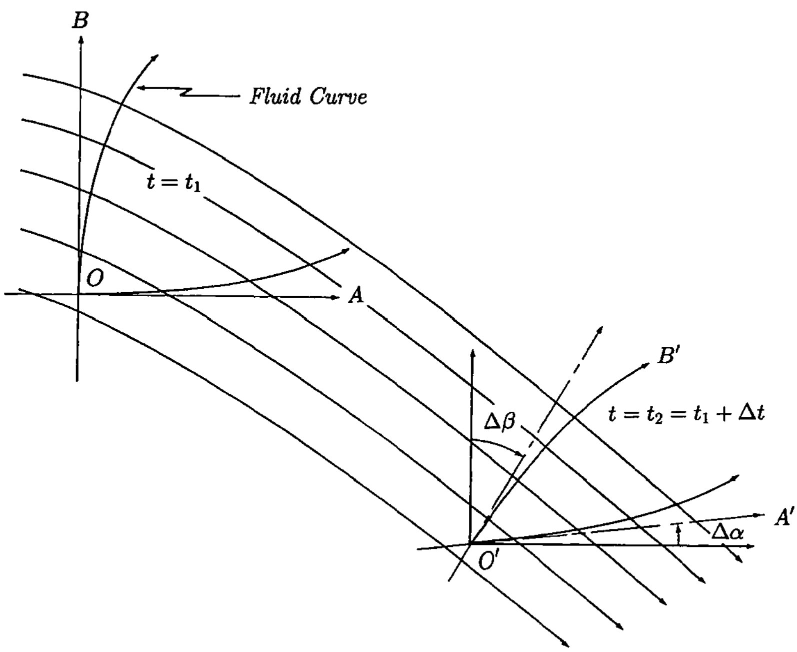

Figure 1–3. Angular rotation and distortion in a fluid.

velocity gradients in the flow. For simplicity consider a two-dimensional fluid element at some initial time t_0 and subsequently at some time $t_0 + \Delta t$. At this subsequent time the fluid element has *in general* undergone translation, rotation and deformation. Figure 1–4 shows the shape of the element at the initial time t_0 and at the subsequent time $t_0 + \Delta t$. From Figure 1–4 the angular displacement of one side of the element is

$$\tan(d\alpha) = \frac{\left(v_0 + \left.\frac{\partial v}{\partial x}\right|_0 dx\right)dt - v_0 dt}{\left(u_0 + \left.\frac{\partial u}{\partial x}\right|_0 dx\right)dt + dx - u_0 dt}$$

which can be written as

$$\tan(d\alpha) = \frac{\left.\frac{\partial v}{\partial x}\right|_0 dt}{\left(1 + \left.\frac{\partial u}{\partial x}\right|_0 dt\right)} \qquad (1-17)$$

Assuming the gradient $\partial u/\partial x|_0$ is at most of order one, then $\partial u/\partial x|_0\, dt << 1$ and is negligible with respect to one. Hence, Eq. (1–17) becomes

$$\tan(d\alpha) = \left.\frac{\partial v}{\partial x}\right|_0 dt$$

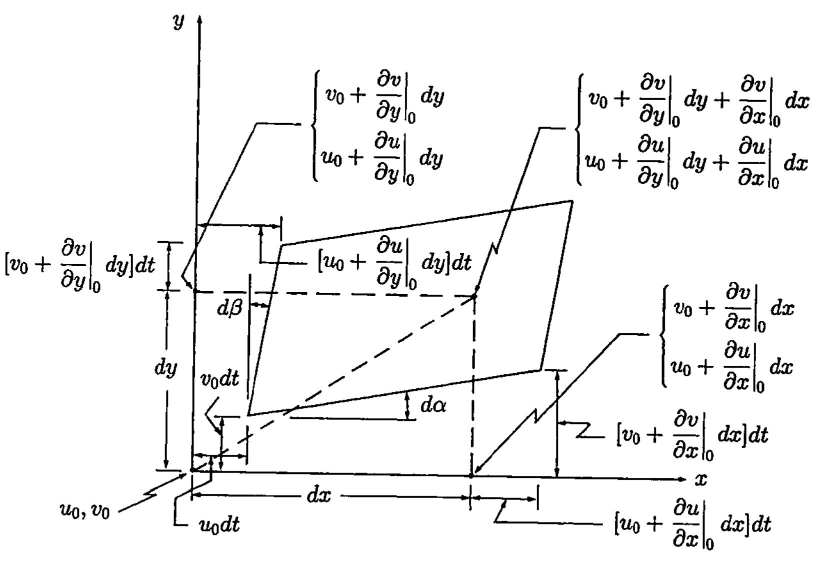

Figure 1–4. Translation, rotation, and deformation of a fluid element.

Further, for small angles $\tan(d\alpha)$ is approximately equal to $d\alpha$. Thus

$$d\alpha = \left.\frac{\partial v}{\partial x}\right|_0 dt \tag{1-18}$$

Similarly, from Figure 1–4 the angular displacement of the other side of the element is

$$\tan(d\beta) \approx d\beta = -\left.\frac{\partial u}{\partial y}\right|_0 dt \tag{1-19}$$

where the negative sign results if a counterclockwise angular displacement is considered positive.

Using these results and Eq. (1–16), we see that the angular velocity depends on the two cross-gradients of the x and y components of velocity, i.e.

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{1-20a}$$

The angular velocity is a vector quantity in the same sense as the translational velocity. Thus, Eq. (1–20a) is taken as the z component of the angular velocity vector. The x and y components are obtained in a similar manner. They are

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \tag{1-20b}$$

and

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \tag{1-20c}$$

The angular velocity vector is written as

$$\vec{\omega} = \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{\imath} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}\right] \tag{1-21}$$

where $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$ are unit vectors in the x, y, z directions, respectively.

For reasons of mathematical simplicity it is preferable to use the vorticity function, ζ, instead of angular velocity. The vorticity vector, $\vec{\zeta}$, is defined as

$$\vec{\zeta} = 2\vec{\omega} \tag{1-22}$$

The other components of the vorticity vector are ξ and η

$$\xi = 2\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \tag{1-23a}$$

$$\eta = 2\omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \tag{1-23b}$$

$$\zeta = 2\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{1-23c}$$