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Introductory concepts

1.1 Some classical analysis

Let Ω be a locally compact, second countable, Hausdorff topological space and let $\mathrm{d}x$ be a Borel measure on Ω. If $1 \leqslant p < \infty$ then we define the real Banach space $L^p(\Omega, \mathrm{d}x)$ to be the vector space of measurable functions $f:\Omega \to \mathbb{R}$ for which the norm

$$\|f\|_p = \left\{ \int_\Omega |f(x)|^p \,\mathrm{d}x \right\}^{1/p}$$

is finite. We identify two functions which coincide outside a null set without further comment. The space $L^\infty(\Omega, \mathrm{d}x)$ is defined to be the space of functions f for which

$$\|f\|_\infty = \min\{\lambda : \mathrm{meas}\{|f| > \lambda\} = 0\}$$

is finite. It may be shown that

$$\|f\|_\infty = \lim_{p \to +\infty} \|f\|_p$$

whenever $f \in L^p$ for all large enough p.

The spaces L^p are all Banach lattices in the sense that

$$\|f\|_p = \| |f| \|_p$$

and

$$\|g\|_p \leqslant \|f\|_p$$

whenever

$$-f \leqslant g \leqslant f.$$

We shall make much use of the ordering of L^p, and introduce the notation

$$(f \wedge g)(x) = \min\{f(x), g(x)\}$$
$$(f \vee g)(x) = \max\{f(x), g(x)\}$$
$$f_+ = f \vee 0, \quad f_- = (-f) \vee 0$$

so that

$$|f| = f_+ + f_-, \quad f = f_+ - f_-.$$

The complex Banach space $L^p_{\mathbb{C}}$ is the algebraic sum $L^p \oplus iL^p$, and the norm is

$$\|f+ig\|_p = \left\{ c\int_\Omega \int_{-\pi}^{\pi} |f\cos\theta + g\sin\theta|^p \,d\theta\,dx \right\}^{1/p}$$
$$= \left\{ c\int_{-\pi}^{\pi} \|f\cos\theta + g\sin\theta\|_p^p \,d\theta \right\}^{1/p} \tag{1.1.1}$$

where

$$c^{-1} = \int_{-\pi}^{\pi} |\cos\theta|^p \,d\theta. \tag{1.1.2}$$

Every bounded linear operator A on $L^p(\Omega)$ extends uniquely to a bounded linear operator $A_{\mathbb{C}}$ on $L^p_{\mathbb{C}}(\Omega)$ which has the same norm and satisfies

$$A_{\mathbb{C}}(\bar{f}) = (A_{\mathbb{C}} f)^-$$

for all $f \in L^p_{\mathbb{C}}(\Omega)$. Although we shall deal mainly with the real Banach spaces, we will use the complex spaces when discussing questions involving analyticity or spectral theory, and will not distinguish between the two notationally.

We will use the following standard results from L^p theory frequently, and without comment.

1.1.1 Duality

If $1 \leqslant p < \infty$ then $(L^p)^*$ can be identified with L^q where

$$1/p + 1/q = 1.$$

1.1.2 Hölder's inequality

Let $f \in L^p$ and $g \in L^q$ where $1 \leqslant p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$. If

$$1/p + 1/q = 1/r$$

and $1 \leqslant r \leqslant \infty$ then $fg \in L^r$ and

$$\|fg\|_r \leqslant \|f\|_p \|g\|_q.$$

1.1.3 The Hausdorff–Young inequality

Let dx be the Lebesgue measure on $\Omega = \mathbb{R}^N$. We define the Fourier transform $\mathscr{F}f$ of the function $f \in L^1 \cap L^2$ by

$$\mathscr{F}f(y) = (2\pi)^{-N/2} \int_{\mathbb{R}^N} f(x) e^{-ix\cdot y} \,dx.$$

If $1 \leqslant p \leqslant 2$ and $p^{-1} + q^{-1} = 1$ then

$$\|\mathscr{F} f\|_q \leqslant (2\pi)^{N/2 - N/p} \|f\|_p$$

for all $f \in L^1 \cap L^2$. Therefore $\mathscr{F}$ may be extended to a bounded linear operator from L^p to L^q for such p.

1.1.4 Young's inequality

Let dx be the Lebesgue measure on $\Omega = \mathbb{R}^N$ and let $f \in L^p$, $g \in L^q$ where $1 \leqslant p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$. If $1 \leqslant r \leqslant \infty$ and

$$1 + 1/r = 1/p + 1/q$$

then the convolution $f * g$ lies in L^r and

$$\|f * g\|_r \leqslant \|f\|_p \|g\|_q.$$

1.1.5 The Riesz–Thorin interpolation theorem

Let $1 \leqslant p_0, p_1, q_0, q_1 \leqslant \infty$ and let A be a linear operator from $L^{p_0} \cap L^{p_1}$ to $L^{q_0} + L^{q_1}$ which satisfies

$$\|Af\|_{q_i} \leqslant M_i \|f\|_{p_i}$$

for all f and $i = 1, 2$. Let $0 < t < 1$ and define p, q by

$$1/p = t/p_1 + (1-t)/p_0, \quad 1/q = t/q_1 + (1-t)/q_0.$$

Then

$$\|Af\|_q \leqslant M_1^t M_0^{1-t} \|f\|_p$$

for all $f \in L^{p_0} \cap L^{p_1}$. Hence A can be extended to a bounded operator from L^p to L^q with norm at most $M_1^t M_0^{1-t}$.

This is a special case of the following.

1.1.6 The Stein interpolation theorem

Let p_i, q_i, p, q, t be as above and let

$$S = \{z \in \mathbb{C} : 0 \leqslant \operatorname{Re} z \leqslant 1\}.$$

Let A_z be linear operators from $L^{p_0} \cap L^{p_1}$ to $L^{q_0} + L^{q_1}$ for all $z \in S$ with the following properties.

(i) $\langle A_z f, g \rangle$ is uniformly bounded and continuous on S and analytic in the interior of S whenever $f \in L^{p_0} \cap L^{p_1}$ and $g \in L^{r_0} \cap L^{r_1}$ where

$$1/q_i + 1/r_i = 1.$$

(ii) $$\|A_{iy} f\|_{q_0} \leqslant M_0 \|f\|_{p_0}$$

for all $f \in L^{p_0} \cap L^{p_1}$ and $y \in \mathbb{R}$.

(iii) $$\| A_{1+iy} f \|_{q_1} \leqslant M_1 \| f \|_{p_1}$$
for all $f \in L^{p_0} \cap L^{p_1}$ and $y \in \mathbb{R}$.

Then
$$\| A_t f \|_q \leqslant M_1^t M_0^{1-t} \| f \|_p$$
for all $f \in L^{p_0} \cap L^{p_1}$. Hence A_t can be extended to a bounded operator from L^p to L^q with norm at most $M_1^t M_0^{1-t}$.

In this theorem the initial domain $L^{p_0} \cap L^{p_1}$ of A_z can be replaced by other domains, such as the set of all functions of the form
$$f = \sum_{r=1}^{n} \alpha_r \chi_{E(r)}$$
where $\alpha_r \in \mathbb{R}$ and $\chi_{E(r)}$ are the characteristic functions of the sets $E(r)$, assumed all to have finite measure.

We now give some standard results and formulae for the Laplace operator acting on $L^2(\mathbb{R}^N)$.

1.1.7 *Domain and spectrum of* $-\Delta$

The Laplacian $H_0 = -\Delta$ is defined on the space $C_c^\infty(\mathbb{R}^N)$ of infinitely differentiable functions of compact support by
$$H_0 f = -\sum_{r=1}^{N} \partial^2 f / \partial x_r^2$$
and satisfies
$$\langle H_0 f, f \rangle = \int_{\mathbb{R}^N} \sum_{r=1}^{N} |\partial f / \partial x_r|^2 \, dx$$
on that domain, so that $H_0 \geqslant 0$. It is known that H_0 is essentially self-adjoint on Schwartz space $\mathscr{S}$, or even on $C_c^\infty(\mathbb{R}^N)$. The spectral resolution of H_0 is achieved using the Fourier transform. Explicitly we have
$$(H_0 f)^\wedge(y) = y^2 \hat{f}(y)$$
on the maximal domain for which the RHS lies in L^2. It follows that the spectrum of H_0 is $[0, \infty)$ and is purely absolutely continuous.

1.1.8 *The heat kernel and Green function*

By the use of Fourier transforms one sees that
$$e^{-H_0 t} f = K_t * f$$
for all $t > 0$ and $f \in L^2(\mathbb{R}^N)$ where
$$K_t(x) = (4\pi t)^{-N/2} e^{-x^2/4t}.$$

Using the formula

$$(H_0+\lambda)^{-1}=\int_0^\infty e^{-H_0 t}e^{-\lambda t}\,dt$$

one deduces that if $\operatorname{Re}\lambda>0$ one has

$$(H_0+\lambda)^{-1}=G_\lambda * f$$

where

$$G_\lambda(x)=\int_0^\infty (4\pi t)^{-N/2}e^{-x^2/4t}e^{-\lambda t}\,dt.$$

The kernel G_λ is strictly positive and becomes infinite as $x\to 0$. It is dominated pointwise by the kernel G_0 of the unbounded operator H_0^{-1}, which is given by

$$\begin{aligned}G_0(x)&=\int_0^\infty (4\pi t)^{-N/2}e^{-x^2/4t}\,dt\\&=c_N|x|^{-(N-2)}\end{aligned}$$

provided $N>2$.

1.1.9 Other functions of the Laplacian

If $F\in L^1(\mathbb{R}^N)$ is a spherically symmetric function then there is a bounded continuous function ϕ on $\mathbb{R}$ such that

$$F*f=\phi(H_0)f$$

for all $f\in L^2$, where $\phi(H_0)$ is defined using the spectral calculus. F and ϕ are related explicitly by

$$\phi(y^2)=\int_{\mathbb{R}^N}F(x)e^{-ix\cdot y}\,dx.$$

This relationship can be extended to other classes of F and ϕ without difficulty.

1.1.10 Minimax

If H is a non-negative self-adjoint operator on a Hilbert space then the spectrum of H is real, and the bottom of the spectrum is given by

$$\begin{aligned}\alpha&=\inf\{\langle Hf,f\rangle : f\in\operatorname{Dom}(H)\text{ and }\|f\|=1\}\\&=\inf\{\|H^{\frac12}f\|^2 : f\in\operatorname{Dom}(H^{\frac12})\text{ and }\|f\|=1\}.\end{aligned}$$

A point λ of the spectrum is said to be in the discrete spectrum if it is isolated *and* of finite multiplicity; otherwise it is said to be in the essential spectrum.

If L is a finite dimensional subspace of $\mathrm{Dom}(H^{\frac{1}{2}})$ we define

$$\lambda_H(L) = \sup\{\|H^{\frac{1}{2}} f\|^2 : f \in L \text{ and } \|f\| = 1\}.$$

We then put

$$E_n = \inf\{\lambda_H(L) : \dim L = n+1\}$$

so that $E_0 = \alpha$ and E_n is an increasing sequence. The minimax theorem gives the following information. The least upper bound of $\{E_n\}$ equals the bottom β of the essential spectrum. The sequence $\{E_n\}$, omitting all values equal to β if there are such, coincides with the discrete spectrum of H in the interval $[\alpha, \beta)$, and each eigenvalue is repeated according to its multiplicity.

1.1.11 Comparison of spectra

If H, K are two self-adjoint operators and $H \leqslant K$ then it is immediate from the definition that

$$\lambda_H(L) \leqslant \lambda_K(L)$$

for all finite-dimensional subspaces L of $\mathrm{Dom}(H^{\frac{1}{2}})$. It follows that

$$E_n(H) \leqslant E_n(K)$$

for all n. The bottoms of the essential spectra of the operators are related by

$$\beta(H) \leqslant \beta(K).$$

1.1.12 The regularised distance function

If Ω is an open set in $\mathbb{R}^N$ then the distance function d on $\mathbb{R}^N$ defined by

$$d(x) = \min\{|x-y| : y \notin \Omega\}$$

is Lipschitz in the sense that

$$|d(x) - d(y)| \leqslant |x - y|$$

but need not be smooth. A theorem of Whitney states that there exists a constant $c > 0$ and a C^∞ function $\tilde{d}:\Omega \to (0, \infty)$ such that

$$c^{-1} d(x) \leqslant \tilde{d}(x) \leqslant c d(x),$$
$$|\nabla \tilde{d}(x)| \leqslant c,$$
$$|\Delta \tilde{d}(x)| \leqslant c \tilde{d}(x)^{-1}$$

for all $x \in \Omega$.

1.2 Quadratic forms

We shall make extensive use of the theory of quadratic forms, and summarise some of the important results.

If $\mathscr{D}$ is a linear subspace of a Hilbert space $\mathscr{H}$ a quadratic form on $\mathscr{D}$ is defined to be a map $Q':\mathscr{D}\times\mathscr{D}\to\mathbb{C}$ such that

(i) $$Q'(\alpha f+\beta g,h)=\alpha Q'(f,h)+\beta Q'(g,h)$$

(ii) $$Q'(h,\alpha f+\beta g)=\bar{\alpha}Q'(h,f)+\bar{\beta}Q'(h,g)$$

(iii) $$Q'(f,g)=\overline{Q'(g,f)}$$

for all $f,g,h\in\mathscr{D}$ and $\alpha,\beta\in\mathbb{C}$. We shall not distinguish between Q' and the map Q from $\mathscr{H}$ to $(-\infty,+\infty]$ defined by

$$Q(f)=\begin{cases}Q'(f,f) & \text{if } f\in\mathscr{D}\\ +\infty & \text{otherwise.}\end{cases}$$

We say that Q is bounded below if there exists $c\in\mathbb{R}$ such that

$$Q(f)\geqslant c\|f\|^2 \tag{1.2.1}$$

for all $f\in\mathscr{H}$, and that Q is non-negative if one may take $c=0$ in (1.2.1). If Q is bounded below, we say that it is closed if for all sequences $f_n\in\mathscr{D}$ such that

$$\lim_{n\to\infty}\|f_n-f\|=0,\qquad \lim_{m,n\to\infty}Q(f_m-f_n)=0$$

it follows that $f\in\mathscr{D}$ and

$$\lim_{n\to\infty}Q(f_n-f)=0.$$

If $H\geqslant 0$ is a self-adjoint operator on a closed linear subspace $\mathscr{L}$ of $\mathscr{H}$ then we define its form by

$$Q(f)=\begin{cases}\langle H^{\frac{1}{2}}f,H^{\frac{1}{2}}f\rangle & \text{if } f\in\operatorname{Dom}(H^{\frac{1}{2}})\\ +\infty & \text{otherwise}\end{cases}$$

and we write $\operatorname{Dom}(Q)$ or $\operatorname{Quad}(H)$ for the domain of $H^{\frac{1}{2}}$ as convenient. We quote the following fundamental result.

Theorem 1.2.1. *If Q is a non-negative form on $\mathscr{H}$ with domain $\mathscr{D}$, then the following conditions are equivalent.*

(i) Q is the form of a self-adjoint operator $H\geqslant 0$ on $\mathscr{L}=\bar{\mathscr{D}}$.
(ii) Q is closed.
(iii) Q is lower semicontinuous as a function from $\mathscr{H}$ to $[0,+\infty]$.

If Q is a non-negative form then its domain is an inner product space with inner product

$$\langle f,g\rangle_Q=\langle f,g\rangle+Q(f,g) \tag{1.2.2}$$

and is complete if and only if Q is closed. A subspace of $\operatorname{Dom}(Q)$ is said to be a form core of Q if it is dense with respect to the norm $\|\ \|_Q$ associated to (1.2.2). Q is said to be closable if it has a closed extension and the closure $\bar{Q}$ is then the least closed extension. It is obvious from Theorem 1.2.1 (iii) that the sum of two closed forms is closed. We shall need two limit theorems.

Theorem 1.2.2. *Let H_n be an increasing sequence of non-negative self-adjoint operators on $\mathscr{H}$. Let Q_n be the associated forms and define Q on $\mathscr{H}$ by*

$$Q(f) = \lim_{n\to\infty} Q_n(f)$$

so that

$$\mathrm{Dom}\,(Q) \subseteq \bigcap_n \mathrm{Quad}\,(H_n).$$

Then there exists a self-adjoint operator $H \geqslant 0$ on the closure $\mathscr{H}_0$ of Dom (Q) *such that*

$$\langle H^{\frac{1}{2}} f, H^{\frac{1}{2}} f \rangle = \lim_{n\to\infty} \langle H_n^{\frac{1}{2}} f, H_n^{\frac{1}{2}} f \rangle$$

for all $f \in \mathrm{Dom}\,(Q)$.

The operator H will be called the form limit of H_n, and we shall mostly use this theorem with $\mathscr{H}_0 = \mathscr{H}$.

Theorem 1.2.3. *Let H_n and H be non-negative self-adjoint operators on $\mathscr{H}$ such that*

$$H_n \geqslant H_{n+1} \geqslant H$$

for all n. Suppose also that the associated forms satisfy

$$\lim_{n\to\infty} Q_n(f) = Q(f)$$

for all f in a form core of Q. Then H_n converges to H in the strong resolvent sense.

We shall use the following notation frequently. If $\mathscr{B}(\Omega)$ is a Banach space of functions on a region (open connected set) Ω in $\mathbb{R}^N$, such as $L^p(\Omega)$, then we write $\mathscr{B}_c(\Omega)$ for the set of f in $\mathscr{B}(\Omega)$ which have compact support, and $\mathscr{B}_0(\Omega)$ for the completion of $\mathscr{B}_c(\Omega)$ in $\mathscr{B}(\Omega)$. We also write $\mathscr{B}_{\mathrm{loc}}(\Omega)$ for the class of functions on Ω which coincide on any compact subset K of Ω with some (K-dependent) element of $\mathscr{B}(\Omega)$.

We shall make particular use of the Sobolev space

$$W^{1,p}(\Omega) = \{ f \in L^p(\Omega) : \nabla f \in L^p(\Omega) \}$$

where ∇f is calculated in the weak sense, and we always assume $1 \leqslant p \leqslant \infty$. It is easy to see that $W^{1,p}(\Omega)$ is a Banach space for an appropriate norm. If $1 \leqslant p < \infty$ we shall take the norm to be

$$\| f \| = (\| f \|_p^p + \| \nabla f \|_p^p)^{1/p} \tag{1.2.3}$$

so that $W^{1,2}(\Omega)$ is a Hilbert space.

Lemma 1.2.4. *If $1 \leqslant p < \infty$ then $W_0^{1,p}(\Omega)$ is the closure in $W^{1,p}(\Omega)$ of $C_c^\infty(\Omega)$.*

Proof. This depends upon a standard mollifier argument which we shall use frequently. Let ϕ be a non-negative C^∞ function on $\mathbb{R}^N$ with support in the unit ball and integral equal to 1. If $f\in W_c^{1,p}(\Omega)$ and $\varepsilon>0$ put $f_\varepsilon=f*\phi_\varepsilon$ where

$$\phi_\varepsilon(x)=\varepsilon^{-N}\phi(x/\varepsilon).$$

It is easy to see that $f_\varepsilon\in C_c^\infty(\Omega)$ for small enough ε and that f_ε converges to f in the norm (1.2.3) as $\varepsilon\to 0$.

We shall be concerned with self-adjoint second order partial differential operators on $L^2(\Omega)$ where Ω is a region in $\mathbb{R}^N$. These operators will be constructed starting from quadratic forms on $C_c^\infty(\Omega)$ of the type

$$Q(f)=\int_\Omega \sum_{i,j} a_{ij}(x)\frac{\partial f}{\partial x_i}\frac{\partial \bar{f}}{\partial x_j}\,\mathrm{d}x \tag{1.2.4}$$

where $a(x)$ is a locally integrable function on Ω with values in the non-negative real symmetric matrices. There are two standard conditions under which Q may be proved to be closable.

Theorem 1.2.5. *If $a\in W_{\mathrm{loc}}^{1,2}(\Omega)$ then Q is closable and the self-adjoint operator H associated with the closure is an extension of the operator*

$$Lf=-\sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial f}{\partial x_j}\right) \tag{1.2.5}$$

defined on $C_c^\infty(\Omega)$.

Proof. The condition on a implies that L maps C_c^∞ into $L^2(\Omega)$ and integration by parts yields the identity

$$\langle Lf,f\rangle=Q(f)$$

for all $f\in C_c^\infty$, so $L\geqslant 0$. We may now apply Davies (1980), Theorem 4.14.

Theorem 1.2.6. *If $a\in L_{\mathrm{loc}}^1(\Omega)$ and $a(x)\geqslant\lambda(x)\,1$ for all $x\in\Omega$ where λ is a strictly positive continuous function, then Q is closable on $C_c^\infty(\Omega)$. If f lies in the domain of the closure then* (1.2.4) *is valid where $\nabla f\in L_{\mathrm{loc}}^2$ is interpreted in the weak sense.*

Proof. Let $\mathscr{D}\subseteq L^2(\Omega)$ consist of all functions which lie in $W_{\mathrm{loc}}^{1,2}(\Omega)$ and for which the integral

$$\tilde{Q}(f)=\int_\Omega \sum_{i,j} a_{ij}(x)\frac{\partial f}{\partial x_i}\frac{\partial \bar{f}}{\partial x_j}\,\mathrm{d}x$$

is finite. A routine calculation shows that $\mathscr{D}$ is complete for the norm

$$|||f|||=(\|f\|_2^2+\tilde{Q}(f))^{\frac{1}{2}}.$$

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Therefore the form $\tilde{Q}$ is complete on $\mathscr{D}$. But Q is a restriction of $\tilde{Q}$ so Q is closable.

If $H \geqslant 0$ is the self-adjoint operator on $L^2(\Omega)$ associated with the closed form Q obtained in Theorem 1.2.5 or 1.2.6, then we say that H satisfies Dirichlet boundary conditions (in the generalised sense). Although one has

$$Hf = -\sum_{i,j} a_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_{i,j}\frac{\partial a_{ij}}{\partial x_i}\frac{\partial f}{\partial x_j} \qquad (1.2.6)$$

in a formal sense, we shall be particularly concerned not to assume that $a_{ij}(x)$ is differentiable. If, however, Ω has smooth boundary and $a_{ij}(x)$ are smooth then the operator H is indeed given by (1.2.6) with Dirichlet boundary conditions in the classical sense.

Henceforth the phrase 'H is an elliptic operator' will mean that H is constructed by the procedure of Theorem 1.2.6 and that

$$\lambda(x)1 \leqslant a(x) \leqslant \mu(x)1 \qquad (1.2.7)$$

where λ and μ are two positive continuous functions on Ω. We will say that H is strictly elliptic if we can take λ to be a positive constant, and uniformly elliptic if we can take λ and μ to be positive constants. For uniformly elliptic operators we have

$$\text{Dom}(\bar{Q}) = \text{Quad}(H) = W_0^{1,2}(\Omega).$$

Theorem 1.2.7. *If H is uniformly elliptic on $L^2(\Omega)$ then $f \in \text{Dom}(H)$ if $f \in W_0^{1,2}(\Omega)$ and there exists $g \in L^2$ such that $Lf = g$ in the sense that*

$$\int_\Omega \sum_{i,j} a_{ij}\frac{\partial f}{\partial x_i}\frac{\partial \bar{u}}{\partial x_j}\,\mathrm{d}x = \int g\bar{u}\,\mathrm{d}x$$

for all $u \in C_c^\infty$. We then have $Hf = g$.

Proof. In abstract terms this is the assertion that $f \in \text{Dom}(H)$ if and only if $f \in \text{Dom}(H^{\frac{1}{2}})$ and there exists $g \in L^2$ such that

$$\langle H^{\frac{1}{2}}f, H^{\frac{1}{2}}u\rangle = \langle g, u\rangle \qquad (1.2.8)$$

for all u in the core C_c^∞ of $\text{Dom}(H^{\frac{1}{2}})$. By taking limits we may assume that (1.2.8) holds for all $u \in \text{Dom}(H^{\frac{1}{2}})$ and then appeal to the spectral theorem as in Davies (1980), Theorem 4.12.

We next give another condition for a non-negative quadratic form to be closable.

Theorem 1.2.8. *Let $\mathscr{D}$ be a dense linear subspace of $\mathscr{H}$ and let*

$$Q(f) = \langle Hf, f\rangle \geqslant 0$$