A NOTE ON H. ISHIHARA AND W. TAKAHASHI MODULUS OF CONVEXITY

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AMS Subject Class. 46B20

1. Introduction:

The classical modulus of convexity of a normed linear space E, introduced by J.A. Clarkson[2] in 1936, is defined as

$$\delta_E(\epsilon)=\inf\{1-\left\|\frac{x+y}{2}\right\| : \|x\|\le 1,\ \|y\|\le 1,\ \|x-y\|\ge\epsilon\} \qquad (0\le\epsilon\le 2)$$

and it is at the origin of a great number of moduli defined since then by several authors[1,3,...,11] up to the present date. One of these is the modulus of convexity of a convex subset C, defined by H. Ishihara and W. Takahashi[7] as

$$\delta(C,\epsilon)=\inf\{1-\frac{1}{r}\left\|z-\frac{x+y}{2}\right\| : x,y,z\in C,\ 0<r\le d(C),\ \|z-x\|\le r,\ \|z-y\|\le r,\ \|x-y\|\ge r\epsilon\}$$

where $0\le\epsilon\le 2$ and d(C) denotes the diameter of C.

We prove in this paper that this modulus does not depend on C but on the affine linear subspace aff(C), spaned by C. Moreover, if $0\in$aff(C) then this modulus coincides with the Clarkson modulus of the linear subspace $\langle C\rangle$, spaned by C.

Throught the paper we shall refer only to convex subsets with more than one point.

2. Results:

It is easy to prove the following properties for every $0\le\epsilon\le 2$:

a) If B_E denotes the unit ball of E then

$$\delta(B_E,\epsilon)=\delta(E,\epsilon)=\delta_E(\epsilon).$$

b) If C and D are convex subsets of E and $D\subset C$ then

$$\delta(C,\epsilon)\le\delta(D,\epsilon).$$

c) For every convex subset $C\subset E$, $x\in E$ and $\lambda\in\mathbb{R}\setminus\{0\}$ it holds

$$\delta(C+x,\epsilon)=\delta(\lambda C,\epsilon)=\delta(C,\epsilon).$$

Lemma. Let C be a convex subset of E and let $x,y,z \in \mathrm{aff}(C)$. Then there exist $\lambda>0$ and $u\in C$ such that $x,y,z \in \lambda(C-u)+u$.

Proof. Let $x,y,z\in \mathrm{aff}(C)$. Then there exist $u_1, \ldots, u_n \in C$ such that $x,y,z\in C^*=\mathrm{aff}(u_1, \ldots, u_n)$. Let u be an interior point of the convex hull of $\{u_1,\ldots,u_n\}$, $Co(u_1,\ldots,u_n)$, with the relative topology of C^*, and let $B(u,s)$ be an open ball of center u and radious s such that $B(u,s)\cap C^*\subset Co(u_1,\ldots,u_n)\subset C$. Then $B(0,s)\cap(C^*-u)\subset C-u$ and $x-u$, $y-u$, $z-u$ are in C^*-u. Let $\lambda>0$ be such that $x-u$, $y-u$, $z-u$ are in $B(0,\lambda s)$. Then $x-u$, $y-u$, $z-u$, are in $\lambda(C-u)$ and therefore, $x,y,z \in \lambda(C-u)+u$.

Proposition. Let C be a convex subset of E. Then $\delta(C,\epsilon)=\delta(\mathrm{aff}(C),\epsilon)$, for every $0\leq\epsilon\leq 2$.

Proof. Let $0\leq\epsilon\leq 2$. From Property (b) it follows $\delta(C,\epsilon)\geq\delta(\mathrm{aff}(C),\epsilon)$. Conversely, let $x,y,z \in \mathrm{aff}(C)$ and $r>0$ be such that $\|z-x\|\leq r$, $\|z-y\|\leq r$ and $\|x-y\|\geq\epsilon r$. From the above lemma there exist $\lambda>0$ and $u\in C$ such that $x,y,z \in \lambda(C-u)+u$. We can also suppose that $\lambda\geq r/d(C)$. So, we have $0<r\leq\lambda d(C)=d(\lambda(C-u)+u)$ and, by Property (c) $1-\frac{1}{r}\left\|z-\frac{x+y}{2}\right\| \geq \delta(\lambda(C-u)+u,\epsilon)=\delta(\lambda(C-u),\epsilon)=\delta(C-u,\epsilon)=\delta(C,\epsilon)$.

Therefore, $\delta(\mathrm{aff}(C),\epsilon)\geq\delta(C,\epsilon)$.

Corollary 1. Let C be a convex subset of E such that $0\in\mathrm{aff}(C)$. Then $\delta(C,\epsilon)=\delta_{<C>}(\epsilon)$, for every $0\leq\epsilon\leq 2$.

Proof. It follows easily from Property (a) and the fact that $\mathrm{aff}(C)=<C>$.

It is also obvious the next corollary.

Corollary 2. If C and D are two convex subsets of E such that either $\mathrm{aff}(C\cap D)=\mathrm{aff}(C)$ or $\mathrm{aff}(C\cap D)=\mathrm{aff}(D)$ then $\delta(D\cap C,\epsilon)=\max\{\delta(C,\epsilon), \delta(D,\epsilon)\}$ for every $0\leq\epsilon\leq 2$.

In [8] the above result is stated without the assumption "$\mathrm{aff}(C\cap D)=\mathrm{aff}(C)$ or $\mathrm{aff}(C\cap D)=\mathrm{aff}(D)$". Unfortunately in this case it is not true. Let, for example, E be the linear space $\mathbb{R}^3$ endowed with a norm whose unit ball is the set $B_E=\{(x,y,z)/ x^2+y^2+z^2\leq 1,\ |z|\leq\frac{1}{2}\}$ and let $a=(0,0,\frac{1}{2})$, $C=B_E+a$ and $D=B_E-a$. Then $\delta_E(1)=0$. On the other hand, the space $<C\cap D>$ is the usual inner product space $(\mathbb{R}^2,\|\ \|_2)$, and and so $\delta_{<C\cap D>}(\epsilon)=1-\sqrt{1-\frac{\epsilon^2}{4}}$ for every $0\leq\epsilon\leq 2$. Therefore,

$$\delta(C,1)=\delta(D,1)=\delta_E(1)=0<1-\sqrt{1-\frac{1}{4}}=\delta_{<C\cap D>}(1)=\delta(C\cap D,1).$$

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A PROPERTY OF NON-STRONGLY REGULAR OPERATORS

by

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Introduction:

Let m be the Lebesgue measure on [0,1]. By L^1 we denote the Banach space of all Lebesgue integrable functions on [0,1] equipped with the norm $\|f\|_1 = \int |f|\,dm$. Let X be a Banach space. In [B_1] J. Bourgain proved that if $T : L^1 \to X$ is a non Dunford - Pettis operator then there exists a Dunford - Pettis operator $D : L^1 \to L^1$ such that the operator $T \circ D$ is not Bochner representable. Recall that on operator from L^1 into a Banach space X is called Dunford - Pettis if it maps weakly compact subsets of L^1 into norm compact subsets of X. (see [B_1], [D-U] for details and undefined notions).

In this paper we prove the following:

Theorem 1: *Let* $T : L^1 \to X$ *be any non strongly regular operator. Then there exists a Dunford - Pettis operator* $D : L^1 \to L^1$ *such that the operator* $T \circ D$ *is not Bochner representable.*

Recall (see [G-G-M-S] Theorem IV. 10) that an operator $T : L^1 \to X$ is strongly regular iff for each $A \subset [0,1]$, $m(A) > 0$ and $\varepsilon > 0$ there is a relatively weakly open subset W of the set $F_A = \{f \in L^1 : f \geq 0, \int f dm = 1, \operatorname{supp} f \subset A\}$ such that diam $(T(W)) < \varepsilon$. It is known ([G-G-M], [Wes]) that any strongly regular operator $T : L^1 \to X$ is Dunford - Pettis.

The proof of Theorem 1 strongly depents on Lemma 2 which is probably of some independent interest. In Lemma 2 we prove an "uncoditional" version of the clasical Theorem of Mazur in the case of a bounded subset of the positive cone of L^1. Lemma 1 is used in the proof of Lemma 2. A consequence of Theorem 1 is following result (Corollary 2): Let K be a closed convex subset of a Banach space X. Suppose that every closed bounded subset of K with the PCP has the RNP. Then if $T: L^1 \to X$ is any non representable "K valued" operator (i.e. $T(F) \subset K$ wheve by F we denote the set of densities in L^1) there exists a Dunfort - Pettis operator $D: L^1 \to L^1$ such that the operator $T \cdot D$ is non representable. Finally we present two examples. Example 1 shows that the D - P operator D in the statement of Theorem 1 can not in general be replaced by a D - P convolution operator.

In Example 2 we construct a non - dentable strongly regular subset K of c_0 with the additional property that for every "K valued" operator $T: L^1 \to c_0$ and for every D-P operator $D: L^1 \to L^1$ the operator $T \cdot D$ is representable.

Remark 1: As noted in [P-S] it follows easily from Proposition VII.4 of [G-G-M-S] that the class of strongly regular operators and the class of representable operators from L^1 to L^1 coincide.

The proof of Corollary 2 shows that the above is true if we replace the target space L^1 by any space X with the property that the PCP and the RNP are equivalent on the subsets of X. (For a simple proof that any strongly regular convolution operator from $L^1(\mathbb{T})$ to $L^1(\mathbb{T})$ is representable see also [P-S]).

Hence it follows from Theorem 1 that for any non representable operator $T: L^1(\mathbb{T}) \to L^1$ there exists a D-P operator $D: L^1(\mathbb{T}) \to L^1(\mathbb{T})$ such that the operator $T \cdot D$ is not representable. This result has been proved in [K - P - R - U] and [P]. Actually it is proved there that the operator D can be taken to be a convolution D - P operator.

Remark 2. (i) One of the first examples of non representable operators from L^1 into C[0,1] is the integral operator $V: L^1 \to C[0,1]$ defined by $Tf(s) = \int_0^s f\,dm$, $f \in L^1$. Bourgain has proved in [B1] that if $D: L^1 \to L^1$ is any D-P operator then the operator $V \cdot D$ is representable.

Hence it follows from Theorem 1 that the operator V is strongly regular. This can also be proved using Theorem IV. 10 of [G-G-M-S].

(ii) In view of Schachermayer's result ([S]) Theorem 1 reduces the KMP versus RNP problem to the following question: Suppose K is a closed convex bounded subset of a Banach space X. Suppose also that K has the KMP. Does this imply that every "K valued" operator $T: L^1 \to X$ has the property that the operator $T \cdot D$ is representable for every D - P operator $D: L^1 \to L^1$?

Lemma 1 : *Let* $G_n \subset L^2$, n=1,2,... *be a sequence of norm compact sets. Suppose that* $\forall g_n \in G_n$, $g_n \to 0$ *weakly. Then there is an increasing* *sequence of positive integers* $n_1 < n_2 < \ldots < n_k < \ldots$ *and a positive constant* M' *such that* $\left\| \sum_{l=1}^{k} \varepsilon_l g_{n_l} \right\|_2 \leq M' \sqrt{k}$ *for all* g_{n_l} *in* G_{n_l} *and for all choices of signs* $\varepsilon_l = \pm 1$, $l = 1,2,\ldots k$.

Proof: Let $M = \sup\{ \|g\|_2, g \in \bigcup_{n \in \mathbb{N}} G_n \}$ Notice that $M < \infty$ otherwise $\forall k \in \mathbb{N}$ $\exists\, n_k$ such that $\| g_{n_k} \|_2 > k$. This is impossible since $g_{n_k} \to 0$ weakly in L_2.

Let h_k, $k=1,2,\ldots$ be an orthonormal basis in L^2. If $g = \sum_{i=1}^{\infty} a_i h_i$ and $k \in \mathbb{N}$ we denote by $g|[1,2,\ldots k]$ the function $\sum_{i=1}^{k} a_i h_i$ and by $g|[k,k+1,\ldots \infty]$ the function $g = \sum_{i=k}^{\infty} a_i h_i$. For $l < m$ $g|[l, m]$ denotes the function $g = \sum_{i=l}^{m} a_i h_i$.

Fix $k \in \mathbb{N}$. Let $a_n = \sup\{\|g|[1,\ldots k]\|_2, g \in G_n\}$. Notice that $a_n \to 0$ since $g_n|[1,\ldots k] \to 0$ for all choices of g_n in G_n.

Using induction we can construct sequences of positive integers $n_1 < n_2 < \ldots < n_s < \ldots$ and $k_1 < k_2 < \ldots < k_s < \ldots$ such that

(i) $\sup\{ \| g|[1, 2, \ldots k_{i-1}] \|_2 , g \in G_{n_i} \} < \dfrac{M}{2^i}$

and

(ii) $\sup \{ \| g|[k_i+1, \ldots \infty] \|_2 , g \in G_{n_1} \cup \ldots \cup G_{n_i} \} < \dfrac{M}{i\, 2^i}$

Notice that if $g_{n_l} \in G_{n_l}$, $\varepsilon_l = \pm 1$

$$\|\sum_{l=1}^{k}\varepsilon_l g_{n_l}\|_2 \le \|\sum_{l=1}^{k}\varepsilon_l (g_{n_l} \mid [k_{l-1}+1,\ldots,k_l])\|_2 + \|\sum_{l=1}^{k}\sum_{\substack{j=1\\ j\neq l}}^{k}(g_{n_j} \mid [k_{l-1}+1,\ldots,k_l])\|_2$$

The functions $g_{n_l}|[k_{l-1}+1, \ldots K_l]$ are disjointly supported and by Bessel's inequality we have that

$$\|\sum_{l=1}^{k}\varepsilon_l (g_{n_l}|[k_{l-1}+1,\ldots,k_l])\|_2 \le M\sqrt{k}$$

On the other hand $\sum_{\substack{j=1\\ j\neq l}}^{k}\|g_{n_j}|[k_{l-1}+1,\ldots,k_l]\|_2 =$

$$\sum_{j=1}^{l-1}\|g_{n_j}[k_{l-1}+1,\ldots,k_l]\|_2 + \sum_{j=l+1}^{k}\|g_{n_j}|[k_{l-1}+1,k_l]\|_2 \le \frac{lM}{12^l} + \frac{M}{2^l}$$

and hence $\|\sum_{l=1}^{k}\sum_{\substack{j=1\\ j\neq l}}^{k} g_{n_j}|[k_{l-1}+1,\ldots k_l]\|_2 \le C$ for some constant C. It is clear now that $\|\sum_{l=1}^{k}\varepsilon_l g_l\|_2 \le M\sqrt{k} + C \le M'\sqrt{k}$ for some constant $M' > 0$, for all choises of signs $\varepsilon_l = \pm 1$, $l = 1,2,\ldots k$.

Lemma 2 : *Let* K *be a bounded subset of the positive cone of* L^1. *Let* f *be a weak limit point of* K. *Then for every* $\varepsilon > 0$ *there exist a finite set* $F = \{f_1,f_2,\ldots f_d\} \subset K$ *and real numbers* a_i, $i = 1,2,\ldots d$, $a_i \ge 0$, $\sum_{i=1}^{d} a_i = 1$ *such that* $\|\sum_{i=1}^{d}\varepsilon_i a_i(f_i - f)\|_1 \le 5\varepsilon$ *for all choices of signs* $\varepsilon_i = \pm 1$, $i = 1,2,\ldots,d$.

Proof : Let M be large enough such that $\int_A f\,dm < \varepsilon$ where $A = \{x \in [0,1] : f(x) > M\}$. Let (h_k) $k = 1,2,\ldots$ be a biorthogonal system in $L^\infty(A^c)$ such that $\|h_k\|_\infty = \|h_k\|_2 = 1$, $\forall k \in \mathbb{N}$. For each $n \in \mathbb{N}$ there exists a finite subset F_n of K, $F_n = \{f_1^n, \ldots, f_{k_n}^n\}$ and real numbers $a_i^n \geq 0$

$i = 1,2, \ldots k_n$, $\sum_{i=1}^{k_n} a_i^n = 1$ such that

$$(1) \quad \sup_{i=1,2,\ldots k_n} \left| \int (f_i^n - f) h_k \, dm \right| < \frac{1}{n} \quad \text{if } k < n.$$

and

$$(2) \quad \left\| \sum_{i=1}^{k_n} a_i^n f_i^n - f \right\| < \frac{1}{n}$$

For $n > \frac{1}{\varepsilon}$ let G_n be the set of all functions g that can be written in the form $g = \sum_{i \in S} a_i^n (f_i^n - f)$ where $\sum_{i \in S} a_i^n \geq \frac{1}{2}$.

We claim that the set $\bigcup_{n > \frac{1}{\varepsilon}} G_n$ is uniformly integrable: If not, there exists a $\vartheta > 0$ such that for all $\delta > 0$ we can find a set $B \subset [0,1]$ with $m(B) < \delta$ and a function $g \in \bigcup_{n > \frac{1}{\varepsilon}} G_n$ such that $\int_B g \, dm > \vartheta$. Choose $\delta_0 > 0$ such that $\int_B f \, dm > \frac{\vartheta}{2}$ for all $B \subset [0,1]$ with $m(B) < \delta_0$.

Assume that the function g is of the form $\sum_{i \in S} a_i^n (f_i^n - f)$ where $\sum_{i \in S} a_i^n \geq \frac{1}{2}$

We have that $\vartheta < \int_B g \, dm = \int_B \sum_{i \in S} a_i^n (f_i^n - f) = \int_B \sum_{i \in S} a_i^n f_i^n - \int_B \sum_{i \in S} a_i^n f \leq$

$\int_B \sum_{i\in S} a_i^n f_i^n \le \int_B \sum_{i=1}^{k_n} a_i^n f_i^n$. On the other hand for $n > \frac{4}{9}$ we have that

$$\left| \int_B \sum_{i=1}^{k_n} a_i^n f_i^n - f \right| \le \left\| \sum_{i=1}^{k_n} a_i^n f_i - f \right\| < \frac{1}{n} < \frac{9}{4} \qquad (2)$$

This implies that $\int_B \sum_{i=1}^{k_n} a_i^n f_i^n < \int_B f + \frac{9}{4} < \frac{9}{2} + \frac{9}{4} = \frac{39}{4}$. This is a contradiction since $\int_B \sum_{i=1}^{k_n} a_i^n f_i^n \ge 9$.

Notice that if for each $n \in \mathbb{N}$ we select a function $g_n \in G_n$ then the sequence $(g_n | A^c)_{n\in\mathbb{N}}$ converges weakly to zero.

Now let $g \in G_n$, and suppose that g can be written in the form $g = \sum_{i\in S} a_i^n (f_i^n - f)$ where $\sum_{i\in S} a_i^n \ge \frac{1}{2}$. Consider the set

$$A_g = \{ x \in [0,1] : \sum_{i\in S} a_i^n (f_i^n(x) > 10M \}$$

We claim that $\int_{A_g\setminus A} |g| < \frac{2}{n}$

To prove the claim notice that $\int_{A_g\setminus A} \left| \sum_{i\in S} a_i^n (f_i^n - f) + \sum_{i\in S^c} a_i^n (f_i^n - f) \right| dm < \frac{1}{n}$

Let $(A_g\setminus A)^- = \{ x \in A_g\setminus A : \sum_{i\in S^c} a_i^n (f_i^n - f)(x) < 0 \}$

and $(A_g\setminus A)^+ = \{ x \in A_g\setminus A : \sum_{i\in S^c} a_i^n (f_i^n - f)(x) \ge 0 \}$.