7

DIAGRAMMATIC METHODS

This chapter is devoted to technicalities related to various expansions already encountered in volume 1, mostly those that derive from the original lattice formulation of the models, be it high or low temperature, strong coupling expansions and to some extent those arising in the guise of Feynman diagrams in the continuous framework. We shall not try to be exhaustive, but rather illustrative, relying on the reader's interest to investigate in greater depth some aspects inadequately treated. Nor shall we try to explore with great sophistication the vast domain of graph theory. There are, however, a number of common features, mostly of topological nature, which we would like to present as examples of the diversity of what looks at first sight like straightforward procedures.

7.1 General Techniques

7.1.1 Definitions and notations

Let a *labelled graph* $\mathcal{G}$ be a collection of v elements from a set of indices, and l pairs of such elements with possible duplications (i.e. multiple links). We shall also interchangeably use the word *diagram* instead of graph. This abstract object is represented by v points (vertices) and l links. Each vertex is labelled by its index.

The problem under consideration will define a set of *admissible graphs*, with a corresponding *weight* $\omega(\mathcal{G})$ (a real or complex number) according to a well-defined set of rules. We wish to find the sum of weights over all admissible graphs.

Possible constraints on the graphs may be

i) the *exclusion* constraint, preventing two vertices from carrying the same index

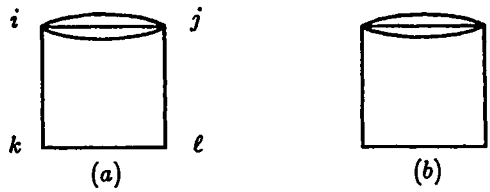


Fig. 1 (*a*) a labelled graph, (*b*) the corresponding free graph.

ii) *simplicity* when two vertices are joined by at most one link (the graph in figure 1(*a*) is not simple).

Take for instance the straightforward high temperature expansion of the Ising partition function

$$\begin{aligned} Z &= 2^{-N} \sum_{\{\sigma_i=\pm1\}} \exp\left(\beta \sum_{\langle ij\rangle} \sigma_i\sigma_j\right) \\ &= 2^{-N} \sum_{\{\sigma_i=\pm1\}} \sum_{\langle ij\rangle} \left(1 + \sum_{n_{ij}=1}^{\infty} \frac{\beta^{n_{ij}}}{n_{ij}!}\sigma_i\sigma_j\right) \end{aligned} \tag{1}$$

Expanding the products, keeping terms with a finite power of β, and averaging over $\sigma_i = \pm 1$, leads to a straightforward high temperature series encountered in volume 1. The successive contributions are associated with graphs defined as follows. A graph has n_{ij} lines joining vertices i and j. Isolated points are not represented as vertices. Since only even powers of σ_i have a nonvanishing unit average, admissible graphs have to obey the following three rules

i) a line can only join vertices indexed by neighbouring sites, and we may think of the graph as drawn on the lattice,
ii) an even number of links are incident on a vertex,
iii) two vertices have distinct labels (the exclusion constraint).

Given an admissible graph, its weight is obtained by associating a factor β to each line, and dividing by the product $\prod_{\langle i,j\rangle} n_{ij}!$ i.e. the order of the symmetry group of the graph under permutation of equivalent links.

We can also write

$$Z = (\cosh\beta)^{Nd} \frac{1}{2^N} \sum_{\{\sigma_i=\pm1\}} \prod_{\langle ij\rangle} (1 + \sigma_i\sigma_j \tanh\beta) \tag{2}$$

which leads for $Z/(\cosh\beta d)^N$ to a different expansion. Admissible graphs are simple with a factor $\tanh\beta$ for each link. Both series are useful in applications.

Two graphs are *isomorphic* when a one-to-one correspondence can be set among vertices and links preserving the incidence relations. The difference lies therefore in the labels of the vertices. Isomorphism leads to equivalence classes called *free graphs* and denoted G. In a pictorial representation, vertices do not carry indices anymore (fig. 1(b)). Conventionally, the corresponding weight $\omega(G)$ will be the average over the equivalent labelled graphs. Call number of configurations $n(G)$ the cardinal of the equivalence class, then

$$\sum_{\mathcal{G}_i \in G} \omega(\mathcal{G}_i) = n(G)\omega(G) \tag{3}$$

This definition is useful whenever the weight of a graph is independent of the labelling of its vertices. In any case, it allows one to disentangle the part $\omega(G)$ that is specific to the model, from the geometry of the lattice, which yields $n(G)$. The following two sections will treat these problems separately.

The above definitions can be extended in various ways.

i) Vertices may be of several types.
ii) Links may have to be oriented.
iii) A generalization may be envisioned, where instead of dealing with 0 and 1 dimensional simplices (vertices and links), one may be required to consider higher dimensional elements (two dimensional plaquettes in the gauge case).
iv) Indices may be compound ones, and links may have to carry indices at their extremities.

This list is of course just indicative of possible extensions.

In some applications, the computation of correlations for instance, a subset of vertices carries fixed indices. Equivalence classes of such graphs will be called *rooted graphs*.

Two vertices x and y on G are *linked* if they can be joined by a path along links of the graph $xz_1, z_1z_2, \ldots, z_ny$. This provides again an equivalence relation on vertices, and the corresponding classes

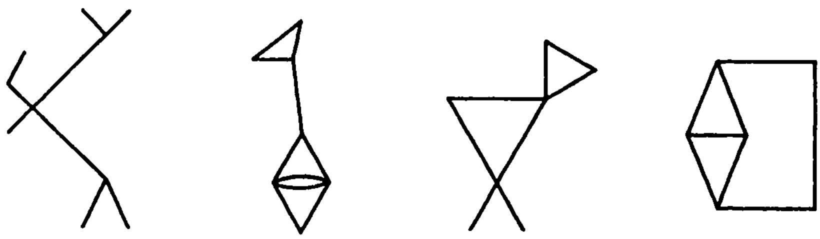


Fig. 2 (*a*) a tree (*b*) a graph with four loops (*c*) a graph with two articulation points (*d*) a multiply connected graph.

define the connected disjoint parts of the graph. A *connected* graph has a unique connected part.

A *cycle* is a closed path of n links, and n vertices, all distinct, starting and ending at the same vertex. A connected graph without cycles is a *tree* (figure 2(a)). The *number of loops* in a connected graph is the minimum number of links which, when removed, leave a tree (figure 2(b)).

An *articulation point* (figure 2(c)) is such that its omission, together with incident links, increases the number of connected parts. It is therefore a vertex which appears on any path linking certain pairs of vertices. In particular, on a tree, all vertices but the external ones (joined to the graph by only one link) are articulation points. A connected graph without articulation points is a *multiply connected* graph: any two vertices belong to a cycle and can therefore be linked by at least two totally distinct paths.

In terms of the following notation
v_k, number of vertices with k incident links
$v = \sum_k v_k$, total number of vertices
l, number of links
b, number of loops
c, number of connected parts
we have the relation

$$2l = \sum_k k v_k, \tag{4}$$

expressing that each link joins two vertices, thus twice the number of links is equal to the sum over vertices weighted by the number of incident links. On the other hand, Euler's relation

$$v + b = c + l \tag{5}$$

follows by successively suppressing the links until only v isolated vertices are left. At each stage, the number of loops decreases or the number of connected parts increases by unity.

i) Compute up to fourth order the Ising partition function on a d-dimensional hypercubic lattice.

Admissible graphs with at most four links are shown on figure 3. The corresponding numbers of configurations on a finite periodic lattice of N sites are respectively Nd, Nd, $Nd(2d-1)$, $\frac{1}{2}Nd(d-1)$ and $\frac{1}{2}Nd(Nd-4d+1)$. Their weights, taking into account the symmetry factor, are respectively $\frac{1}{2}\beta^2$, $\frac{1}{24}\beta^4$, $\frac{1}{4}\beta^4$, β^4, $\frac{1}{4}\beta^4$. Summing these contributions yields

$$Z = 1 + \tfrac{1}{2}Nd\beta^2 + \left[\tfrac{1}{12}Nd(6d-7) + \tfrac{1}{8}N^2d^2\right]\beta^4 + \mathcal{O}(\beta^6) \tag{6}$$

To this order, one verifies the extensive character of the free energy. Indeed

$$-\frac{\beta\mathcal{F}}{N} = \frac{F}{N} = \frac{\ln Z}{N} = \tfrac{1}{2}d\beta^2 + \tfrac{1}{12}d(6d-7)\beta^4 + \mathcal{O}(\beta^6) \tag{7}$$

is N-independent. The notation $\mathcal{F}$ refers to the traditional thermodynamic definition.

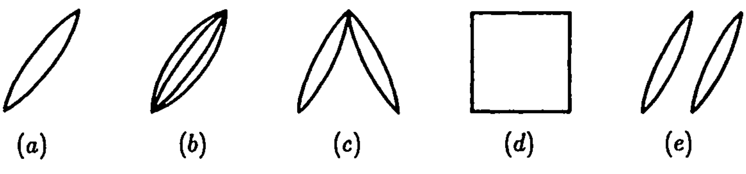


Fig. 3 Graphs for the Ising model up to order β^4.

This example illustrates the convenience of using $\tanh\beta$ rather than β as a small parameter in a high temperature expansion. Indeed, the corresponding graphs have now to satisfy the constraint of simplicity. Only the graph of figure 3(d) leads to a nonvanishing contribution in an expansion up to order $\tanh\beta^4$, and we find

$$Z = (\cosh\beta)^{Nd}[1 + \tfrac{1}{2}Nd(d-1)\tanh^4\beta + \mathcal{O}(\tanh^6\beta)] \tag{8}$$

Equations (6) and (8) are easily shown to agree.

This simplification is useful in avoiding the proliferation of graphs, when investigating the higher order contributions. Other techniques will be discussed below.

ii) *Kirchoff's theorem.* The previous definitions allow us to recall a theorem due to Kirchoff giving in closed form the number of distinct trees that can be drawn on a connected graph, which uses the same set of vertices, the so called *spanning trees.* To a connected graph G we associate the incidence matrix which is a topological equivalent of (minus) the Laplacian. Along the main diagonal, $(-\Delta)_{ii}$ is equal to the number of links incident on vertex i, while $(-\Delta)_{ij}$ for distinct i and j is minus the number of links joining the vertices i and j. Elements in each line or column of $-\Delta$ add to zero, and therefore $\det(-\Delta)$ is zero, corresponding to the existence of a unique zero mode, a constant, since the graph is connected. The claim is that the determinant of any principal minor ($(-1)^{i+j}$ times the determinant obtained by deleting the ith line and jth column) is equal to the number of spanning trees. From the fact that the graph is connected, there exists a unique vector (the zero mode) with equal components corresponding to the zero eigenvalue of $-\Delta$ up to an overall factor. Let M_{ij} be the principal minor of the element ij (including the sign). From

$$\sum_j (-\Delta)_{ij} M_{kj} = \delta_{ik} \det(-\Delta) = 0$$

it follows that, for fixed k, all M_{kj} are equal, and since the matrix is symmetric as is $-\Delta$, all M_{kj} are equal to the same value M. It is therefore sufficient to compute $M = M_{11}$, the determinant of the matrix $-\Delta$ with the first line and column deleted. Let v denote the number of vertices, and $\ell \geq v-1$ the number of links. Define a $l \times v$ matrix $L_{\alpha i}$ where α labels links, i vertices, after giving to each link an arbitrary orientation, through

$$L_{\alpha i} = \begin{cases} +1 & \text{if the link } \alpha\text{, incident on } i \text{ is oriented off } i \\ -1 & \text{if the link } \alpha\text{, incident on } i \text{ is oriented towards } i \\ 0 & \text{if the link } \alpha \text{ is not incident on } i \end{cases}$$

Then $(-\Delta) = L^{\mathrm{T}} L$. Call L' the matrix L with its first column deleted, in such a way that

$$M = \det L'^{\mathrm{T}} L' = \sum_{\{\alpha_2 < \cdots < \alpha_v\}} \det L'^{\mathrm{T}}_{\alpha_2 \ldots \alpha_v} L'_{\alpha_2 \ldots \alpha_v}$$

with the sum running over all $(v-1)\times(v-1)$ matrices $L'_{\alpha_2\ldots\alpha_v}$ obtained by selecting $v-1$ rows in L' labelled $\alpha_2 < \alpha_3 < \cdots < \alpha_v$. The above equality is a classical identity in the theory of determinants. Each term in the sum is of the form $(\det L'_{\alpha_2\ldots\alpha_v})^2$, and is zero unless the map $i \to \alpha_i$ assigns to each vertex $i = 2,\ldots, v$ an incident link, in which case the matrix $L'_{\alpha_2\ldots\alpha_v}$ differs from a permutation matrix only by the fact that its entries are ± 1 instead of $+1$. This has two consequences. The first is that $(\det L'_{\alpha_2\ldots\alpha_v})^2$ is equal to unity, and the second that it is in one-to-one correspondence with a spanning tree. This proves Kirchoff's theorem. It gives a topological meaning to the Laplacian which proves useful in percolation and polymer problems. We shall encounter an application in chapter 11.

7.1.2 Connected graphs and cumulants

The fundamental property of exponentiation relies on the following conditions.

i) The empty graph (no vertex, no link) is admissible, with a weight equal to 1. It has no connected parts ($c = 0$) and is therefore not connected ($c \neq 1$).
ii) Every union of admissible graphs is admissible.
iii) The weight of a graph factors into contributions from its connected parts.

If these hypotheses are valid, the sum over all admissible graphs is equal to the exponential of the sum over connected graphs. The exclusion constraint is not compatible with condition (ii). The reader will check on the above example of the β-expansion up to β^4 for the Ising model, that the preceding property is wrong, namely the free energy is not directly given in terms of contributions from connected graphs. It is our present task to modify the rules in order to find a direct expansion for the free energy.

The proof is quite simple. An arbitrary graph is built by drawing successively and independently its c connected parts $\mathcal{G}_1,\ldots\mathcal{G}_c$. The order being immaterial, each disconnected graph is therefore obtained $c!$ times instead of one. Using factorization of weights and summing on c, one finds

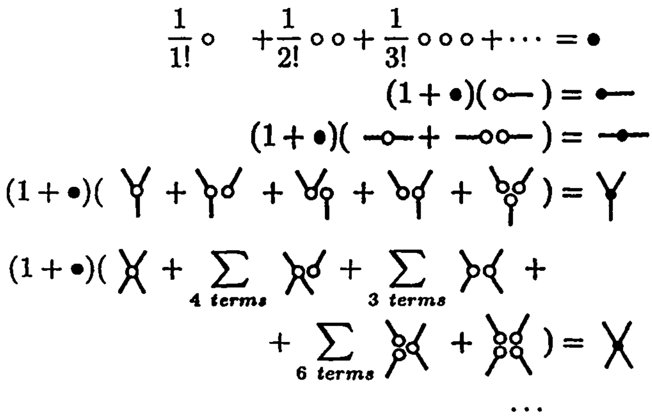
$$\sum_{\mathcal{G}} \omega(\mathcal{G}) = \sum_{c=0}^{\infty} \frac{1}{c!} \sum_{\substack{\mathcal{G}_1,\ldots,\mathcal{G}_c \\ \text{connected}}} \omega(\mathcal{G}_1)\cdots\omega(\mathcal{G}_c) \tag{9}$$

On the right-hand side, one recognizes the expansion of an exponential, namely

$$\sum_{\mathcal{G}} \omega(\mathcal{G}) = \exp\left(\sum_{\mathcal{G}\ \text{connected}} \omega(\mathcal{G}) \right) \tag{10}$$

The above exponential property is crucial in the computation of extensive quantities as well as in the study of various asymptotic behaviours, correlation lengths, or boundary effects. Unfortunately the exclusion rule forbids its direct application. We will now modify these rules, using the *cumulant* method, to restore exponentiation.

Assume that another set of rules can be found avoiding the exclusion constraint. To distinguish them, we represent vertices of the new graphs by open instead of full circles. A contribution of a new graph represents part of the former ones, obtained by identifying vertices with the same labels. If one requires that the new expansion reproduces the previous results, one obtains a set of equations, written symbolically as



(11)

The factor $(1 + \bullet)$ follows from the possibility of identifying as many isolated vertices as one may wish. In this relation, all vertices carry the same index, which has been omitted for clarity.

To solve these equations, one must of course define the associated weights. Take the Ising case as a typical example. In the standard expansion which enforces the exclusion constraint, we have

i) a factor z_k for a vertex with k incident links

ii) a factor β for each link
iii) the product of these factors is divided by the order of the symmetry group of the graph.

The expansion without the exclusion constraint follows from similar rules, except for the fact that the factors z_k are to be replaced by cumulants u_k. The equations (11) read explicitly

$$\begin{aligned} e^{u_0} - 1 &= z_0 - 1 \\ u_1 &= z_1/z_0 \\ u_2 + u_1^2 &= z_2/z_0 \\ u_3 + 3u_2u_1 + u_1^3 &= z_3/z_0 \\ u_4 + 4u_3u_1 + 3u_2^2 + 6u_2u_1^2 + u_1^4 &= z_4/z_0 \\ &\cdots \end{aligned} \tag{12}$$

The cumulants u_k are obtained by inverting this system as

$$\begin{aligned} u_0 &= \ln z_0 \\ u_1 &= \frac{z_1}{z_0} \\ u_2 &= \frac{z_2}{z_0} - \left(\frac{z_1}{z_0}\right)^2 \\ u_3 &= \frac{z_3}{z_0} - 3\frac{z_2}{z_0}\frac{z_1}{z_0} + 2\left(\frac{z_1}{z_0}\right)^3 \\ u_4 &= \frac{z_4}{z_0} - 4\frac{z_3}{z_0}\frac{z_1}{z_0} - 3\left(\frac{z_2}{z_0}\right)^2 + 12\frac{z_2}{z_0}\left(\frac{z_1}{z_0}\right)^2 - 6\left(\frac{z_1}{z_0}\right)^4 \\ &\cdots \end{aligned} \tag{13}$$

To obtain a compact form, write $z(h)$ and $u(h)$ for the generating functions

$$z(h) = \sum_{k=0}^{\infty} \frac{z_k}{k!} h^k \tag{14}$$

$$u(h) = \sum_{k=0}^{\infty} \frac{u_k}{k!} h^k \tag{15}$$

The above relations are then simply

$$u(h) = \ln z(h) \tag{16}$$

i) Justify in more detail the cumulant procedure and check equation (16) up to order four against (13).
ii) Rederive the high temperature expansion in the Ising case using cumulants. The initial rules assume $z_{2k} = 1$, $z_{2k+1} = 0$, thus

$$z(h) = \cosh h \tag{17}$$
$$u(h) = \ln\cosh h = \tfrac{1}{2}h^2 - \tfrac{1}{12}h^4 + \cdots \tag{18}$$

The connected graphs are the first four on figure 3 up to fourth order. Since the exclusion rule no longer applies, the configuration numbers are modified to Nd, Nd, $\frac{1}{2}N(2d)^2$, $\frac{3}{4}Nd(2d-1)$ respectively. These numbers are all proportional to the number of sites N, because of translational invariance, and this insures that the free energy is extensive. Since $u_2 = 1$, $u_4 = -2$, the new weights are $\frac{1}{2}\beta^2$, $\frac{1}{6}\beta^4$, $-\frac{1}{2}\beta^4, \beta^4$. The weights are no longer positive, reflecting a similar loss of positivity of the cumulants. Summation over connected graphs yields immediately formula (7) and confirms the general property. The same method applies to the expansion in $\tanh\beta$. One has to introduce factors z_k which depend on the relative direction of incident links, and the graphs obtained in this way are no longer simple, which limits the interest of the method. The choice between simplicity and connectivity depends on the problem at hand. For gauge theories, we shall see that simplicity is more advantageous.

For rooted graphs, the distribution of fixed indices over connected parts leads to equations generalizing (11). Let $Z\langle i_1 \cdots i_k\rangle$ (Z is the partition function) denote the sum over all graphs (connected or disconnected) with roots $i_1, \ldots, i_k$, and $\langle i_1 \cdots i_k\rangle_c$ the sum over connected graphs using cumulants, one has in general

$$\begin{aligned}
<i_1> &= <i_1>_c \\
<i_1 i_2> &= <i_1 i_2>_c + <i_1>_c <i_2>_c \\
<i_1 i_2 i_3> &= <i_1 i_2 i_3>_c + <i_1 i_2>_c <i_3>_c + <i_1 i_3>_c <i_2>_c \\
&\quad + <i_2 i_3>_c <i_1>_c + <i_1>_c <i_2>_c <i_3>_c \\
&\cdots
\end{aligned} \tag{19}$$

Disconnected graphs factorize partly into connected parts involving no roots, the sum of which yields the factor Z, and partly into connected rooted graphs which will realize all possible partitions of the set of indices $i_1, \ldots, i_k$. These properties become more