Uniqueness of sporadic groups

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Initial work on the sporadic finite simple groups falls into one or more of the following categories:

Discovery
Structure
Existence
Uniqueness

More precisely let $\mathcal{H}$ be some group theoretic hypothesis. A group theorist begins to investigate groups G satisfying $\mathcal{H}$ and generates information about the structure of such groups. Typical examples of structural information include the group order, the isomorphism type of normalizers of subgroups of prime order, and perhaps eventually the character table of G. When a sufficiently large body of self-consistent structural information has been generated, the group is said to be *discovered*. This is roughly the point where the group theoretic community first becomes convinced that the group exists.

The group actually *exists* when there is a proof that there is at least one group satisfying hypothesis $\mathcal{H}$, while the group is *unique* when there is a proof that, up to isomorphism, there is at most one group satisfying $\mathcal{H}$. More detailed information about the group structure usually comes later and might include the calculation of the automorphism group and Schur multiplier of G, an enumeration of the maximal subgroups of G, and the generation of the modular character tables for G.

As part of the ongoing effort to produce a complete, unified, and accessible proof of the Classification Theorem, Aschbacher has begun to try to write down in one place a complete and fairly self-contained proof that the 26 sporadic groups exist and are unique. The plan is to generate at the same time the basic structural information about each sporadic group necessary for the Classification. This program dovetails with the Gorenstein-Lyons-Solomon effort to "revise" the proof of the Classification, since GLS give themselves the existence, uniqueness, and basic structure of each sporadic group.

This article concerns itself only with the uniqueness question. The first part consists of an exposition of machinery developed in [**2**] to deal with the

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uniqueness of some of the sporadic groups. To understand and appreciate the statement of the main results from [**2**] it is first necessary to introduce some graph theoretic and geometric concepts. This is done in section 2. Then the main theorems from [**2**] are stated in section 3, where there is also a very brief discussion of the proof of these results. Section 4 is devoted to a discussion of how to use the theory in [**2**] to prove the uniqueness of some of the larger sporadic groups via their local geometries. The final section contains some speculation about possible ways to establish the uniqueness of each of the sporadic groups and the diff iculties involved in such an undertaking.

A final general remark. It seems to us that any good second generation treatment of the uniqueness of the sporadic groups must do several things. It must be simple, clear, and elegant. It should be independent of machine calculation. Finally it should be as uniform as possible with a minimum of case analysis. Given our present understanding of the sporadic groups as 26 independent entities, some amount of case analysis seems unavoidable, but the machinery in [**2**] gives hope that some differences can be minimized. However the theory in [**2**] is in its infancy and much remains to be done before a truly uniform treatment of the uniqueness of the sporadic groups exists.

Section 2. Graphs.

In this section Δ is a graph. Let x be a vertex in Δ, write $\Delta(x)$ for the set of vertices distinct from x and adjacent to x in Δ, $x^{\perp} = \Delta(x) \cup \{x\}$, and $\Delta^n(x)$ for the set of vertices at distance n from x. A *morphism* $d : \Delta :\rightarrow \Delta'$ of graphs is a map of vertices such that $d(x^{\perp}) \subseteq d(x)^{\perp}$ for all $x \in \Delta$.

Let $P = P(\Delta)$ be the set of paths in Δ. Thus the members of P are the finite sequences $p = x_0 \cdots x_r$ from Δ with $x_{i+1} \in x_i^{\perp}$ for all i. Write $org(p)$, $end(p)$ for the *origin* x_0 and *end* x_r of p, respectively. Write pq for the concatenation of paths p and q such that $end(p) = org(q)$. Write p^{-1} for the path $x_r \cdots x_0$. The path $p = x_0 \cdots x_r$ is a *circuit* if $x_r = x_0$.

Define an equivalence relation $\sim$ on P to be *P-invariant* if the following four conditions are satisfied:

(PI1) If $p \sim q$ then $org(p) = org(q)$ and $end(p) = end(q)$.
(PI2) $rr^{-1} \sim org(r)$ for all $r \in P$.
(PI3) Whenever $p \sim p'$ and $q \sim q'$ with $end(p) = org(q)$, then also $pq \sim p'q'$.
(PI4) $x \sim xx$ for all $x \in \Delta$.

Define the *kernel* of an equivalence relation $\sim$ on P to be the set $ker(\sim)$ of all circuits s such that $s \sim org(s)$. Define a subset S of P to be *closed* if it is the kernel of some invariant equivalence relation and define the *closure* of a set T of circuits to be the intersection of all closed subsets containing T. There is an intrinsic characterization of closed sets in section 2 of [**2**] which shows the intersection of closed sets is closed, so the closure of T is well-defined.

Given a set S of circuits of Δ, define a relation $\sim_S$ on P by $p \sim_S q$ if p and q have the same origin and end and $pq^{-1} \in S$. It is easy to check that:

(2.1) *Let $\sim$ be a P-invariant equivalence relation. Then $\sim = \sim_{ker(\sim)}$. In particular a set S of circuits of Δ is closed if and only if $\sim_S$ is a P-invariant equivalence relation on P.*

Define the *basic relation* to be the relation $\sim_{Bas}$ where Bas is the smallest closed subset of P. Write $\equiv$ for $\sim_{Bas}$. Notice $\equiv$ is characterized by the property that if $\sim$ is P-invariant and $a \equiv b$ then $a \sim b$.

Write $[P]$ for the set of equivalence classes $[p]$ of the basic relation $\equiv$. For $x \in \Delta$, write $\Sigma(\Delta, x)$ for the set of paths p with origin x and write $\pi_1(\Delta, x)$ for the set of classes $[p] \in [P]$ with p a circuit and $org(p) = x$. As $\equiv$ is P-invariant, $\pi_1(\Delta, x)$ is a group under the product $[p][q] = [pq]$. Of course $\pi_1(\Delta, x)$ is the fundamental group of the graph and is free (*cf.* Section 5.1 in Serre [**12**]), but we won't need this fact.

Define Δ to be *r-generated* if the closure of the set of all circuits of length at most r is the set of all circuits. We say Δ is *triangulable* if Δ is 3-generated. Intuitively Δ is triangulable if each circuit is the product of triangles, and a given path can usually be seen to be in the closure of the triangles by drawing suitable pictures like those suggested by 2.3 below. More formally:

(2.2) *Δ is triangulable if and only if for each $x \in \Delta$, $\pi_1(\Delta, x)$ is generated by classes $[rtr^{-1}]$, $r \in P$, t triangle, $org(r) = x$, $end(r) = org(t)$.*

Define a morphism $d : \Gamma \to \Delta$ of graphs to be a *local bijection* if for all $\alpha \in \Gamma$,

$$d_\alpha = d|_{\alpha^\perp} : \alpha^\perp \to d(\alpha)^\perp$$

is a bijection. Define d to be a *fibering* if d is a surjective local bijection. The fibering is *connected* if its domain Γ is connected. The fibering is a *covering* if $d_\alpha : \alpha^\perp \to d(\alpha)^\perp$ is an isomorphism for all $\alpha \in \Gamma$. We say Δ is *simply connected* if Δ is connected and Δ possesses no proper connected coverings.

Caution. In the combinatorial group theoretic literature the term "covering" is sometimes used as we use the term fibering. However we prefer to reserve the word covering for a local isomorphism. For example coverings of topological spaces and Tits' coverings of geometries in [**16**] are local isomorphisms.

Given a P-invariant equivalence relation $\sim$, define $P/\sim = \tilde{P}$ to be the set of equivalence classes of $\sim$ and make $\tilde{P}$ into a graph by decreeing that $\tilde{p}$ is adjacent to $\tilde{q}$ if $p \sim qx$, where $x = end(p) \in \Delta(end(q))$. Notice that if $p \sim qx$ then $q \sim p \cdot end(q)$, so our graph is undirected.

Recall $\Sigma(\Delta, x)$ denotes the set of paths with origin x. Write $\Sigma(\Delta, x)/\sim$ for the set of classes $\tilde{p}$ with $p \in \Sigma(\Delta, x)$ and $\tilde{\pi}_1(\Delta, x)$ for the group of all $\tilde{p}$ with $p \in \Sigma(\Delta, x)$ a circuit.

(2.3) *Assume Δ is connected and let T be the closure of the set of triangles of Δ. Then*

(1) $end : \Sigma(\Delta, x)/\equiv \to \Delta$ is a universal connected fibering for Δ.

(2) $end : \Sigma(\Delta, x)/\sim_T \to \Delta$ is a universal connected covering for Δ.

(3) Δ is simply connected if and only if Δ is triangulable.

It will be important for us to know when certain graphs are simply connected. Lemma 2.3.3 says Δ is simply connected if and only if it is triangulable, and in the graphs we encounter this turns out to be an effective means for proving simple connectivity.

Remark. Let $K = K(\Delta)$ be the simplicial complex whose vertices are the complete subgraphs of Δ. At this conference Tits asked if Δ is simply connected if and only if t he topological space $|K|$ of K is simply connected. The answer is yes. Namely if $\sim = \sim_T$ where T is the closure of all triangles, then $\tilde{\pi}_1(\Delta, x)$ is the *edge path group* of K (*cf.* Chapter 3, Section 6 of [**14**]) so by Theorem 3.6.16 in [**14**] , $\tilde{\pi}_1(\Delta, x) \cong \pi_1(|K|, x)$, the fundamental group of $|K|$.

We close the section with a few elementary lemmas from [**2**] on triangulation. In each case S is a closed subset of P.

(2.4) *(1) If pq, pr, and $r^{-1}q$ are circuits with pr, $r^{-1}q \in S$, then $pq \in S$.*

(2) Let $a_i, b_i, c_j \in P$, $1 \le i \le n$, $1 \le j < n$, such that $org(a_i) = x$, $end(b_i) = u$, $end(a_i) = org(b_i) = org(c_i) = end(c_{i-1})$ for $1 \le i \le n$. Assume $a_i c_i a_{i+1}^{-1}$ and $b_i^{-1} c_i b_{i+1}$ are in S for $1 \le i < n$. Then $a_n b_n b_1^{-1} a_1^{-1} \in S$.

Given integers n, m with $n \ge 2$, define $|m|_n = r$, where $0 \le r \le n/2$ and $m \equiv r$ or $-r \mod n$. Then define a circuit $p = x_0 \cdots x_n$ of length n to be a *n-gon* if $d(x_i, x_j) = |i - j|_n$, for all i, j, $0 \le i, j \le n$. Define $gon(S)$ to be the least r for which there exists an *S-nontrivial circuit* (*i.e.* a circuit not in S) of length r.

(2.5) *Let $r = gon(S)$ and p be an S-nontrivial circuit of length r. Then p is an r-gon.*

To show S consists of all circuits, by 2.5 it suffices to show that for all $r \leq 2diam(\Delta) + 1$, each r-gon is in S.

(2.6) *Assume $r = gon(S) > 3$ and for each $x \in \Delta$ and $u \in \Delta^2(x)$, $\Delta(x, u)$ is connected. Then $gon(S) > 4$.*

PROOF: By 2.5 we may assume $p = x_0 \ldots x_4$ is an S-nontrivial square. By hypothesis there is a path $x_1 = y_1, \ldots, y_n = x_3$ in $\Delta(x_0, x_3)$. Now appeal to 2.4.2 with $a_i = x_0 y_i$, $b_i = y_i x_2$, and $c_i = y_i y_{i+1}$.

Section 3. The Main Theorems of [**2**].

Define a *uniqueness system* to be a 4-tuple $\mathcal{U} = (G, H, \Delta, \Delta_H)$ such that G is an edge transitive group of automorphisms of the undirected graph Δ, $H \leq G$, Δ_H is a graph with vertex set xH and edge set $(x, y)H$ for some $x \in \Delta$ and $y \in \Delta(x) \cap xH$, and:

(U) $G = \langle H, G_x \rangle$, $G_x = \langle G_{x,y}, H_x \rangle$, and $H = \langle H(\{x, y\}), H_x \rangle$.

Say Δ_H is a *base* for $\mathcal{U}$ if the closure of the G-conjugates of circuits in Δ_H is the set of all circuits of Δ.

Define a *similarity* of uniqueness systems $\mathcal{U}, \bar{\mathcal{U}}$ to be a pair of isomorphims $\alpha : G_x \to \bar{G}_{\bar{x}}$ and $\zeta : H \to \bar{H}$ such that $\alpha = \zeta$ on H_x, $H_x \zeta = \bar{H}_{\bar{x}}$, $G_{x,y}\alpha = \bar{G}_{\bar{x},\bar{y}}$, and $H(\{x, y\})\zeta = \bar{H}(\{\bar{x}, \bar{y}\})$ for some edges $(x, y), (\bar{x}, \bar{y})$ of $\Delta_H, \Delta_{\bar{H}}$, respectively. We say the similarity is *with respect to* $(x, y), (\bar{x}, \bar{y})$ if we wish to emphasize the role of those edges. The similarity is an *equivalence* if there exists $t \in H$ with cycle (x, y) such that $(b^t)\alpha = (b\alpha)^{t\zeta}$ for all $b \in G_{x,y}$.

Define a *morphism* of uniqueness systems $\mathcal{U}, \bar{\mathcal{U}}$ to be a gro up homomorphism $d : G \to \bar{G}$ such that the restrictions $d : H \to \bar{H}$ and $d : G_x \to \bar{G}_{\bar{x}}$ are isomorphisms defining a similarity of $\mathcal{U}$ with $\bar{\mathcal{U}}$. Notice d induces a map $d : \Delta \to \bar{\Delta}$ defined by $(xg)d = \bar{x}(gd)$; it turns out this map is a fibering and induces an isomorphism $d : \Delta_H \to \bar{\Delta}_{\bar{H}}$.

We are now in a position to state the principal results of [**2**]. The Main Theorem is:

THEOREM 1. *Assume $\mathcal{U}, \bar{\mathcal{U}}$ are equivalent uniqueness systems such that $\Delta_H, \bar{\Delta}_{\bar{H}}$ are bases for $\Delta, \bar{\Delta}$, respectively. Then $\mathcal{U} \cong \bar{\mathcal{U}}$.*

COROLLARY. *Assume $\mathcal{U}$ and $\bar{\mathcal{U}}$ are equivalent uniqueness systems, Δ is triangulable, each triangle of Δ is G-conjugate to a triangle of Δ_H, and $\bar{\mathcal{U}}$ also satisfies these hypotheses. Then $\mathcal{U} \cong \bar{\mathcal{U}}$.*

In order to apply Theorem 1 and its Corollary, we need effective means for verifying the equivalence of uniqueness systems. Several such results are contained in [**2**]; we record two of them as typical:

THEOREM 2. *Assume $\mathcal{U}$ and $\bar{\mathcal{U}}$ are similar uniqueness systems and for some edge (x, y) of Δ_H, $Aut(G_{x,y}) \cap C(H_{x,y}) = 1$. Then $\mathcal{U}$ is equivalent to $\bar{\mathcal{U}}$.*

THEOREM 3. *Assume $\mathcal{U}, \bar{\mathcal{U}}$ are uniqueness systems satisfying Hypothesis V below with respect to edges $(x, y), (\bar{x}, \bar{y})$ and $\alpha : G_x \to \bar{G}_{\bar{x}}$ and $\zeta : H \to \bar{H}$ are isomorphisms such that $G_{x,y}\alpha = \bar{G}_{\bar{x},\bar{y}}$, $H_x\zeta = \bar{H}_{\bar{x}} = H_x\alpha$, and $H(\{x, y\})\zeta = \bar{H}(\{\bar{x}, \bar{y}\})$. Then $\mathcal{U}$ and $\bar{\mathcal{U}}$ are similar.*

HYPOTHESIS V. *The uniqueness system $\mathcal{U} = (G, H, \Delta, \Delta_H)$ satisfies the following four conditions for some edge (x, y) of Δ_H:*
(V1) $Aut(H_x) = Aut_{Aut(H)}(H_x)Aut_{Aut(G_x)}(H_x)$.
(V2) $N_{Aut(G_x)}(H_x) \leq N(G_{x,y}^{H_x})C(H_x)$.
(V3) $N_{Aut(H)}(H_x) \leq N(H_xH(\{x, y\})H_x)C(H_x)$.
(V4) $N_{H_x}(H_{x,y}) \leq N_{G_x}(G_{x,y})$.

We close this section with a brief discussion of the proof of Theorem 1.

Let $I = \{1, \ldots, n\}$ be a set of finite order n. Recall an *amalgam of rank* n is a family

$$A = (\alpha_{J,K} : P_J \to P_K : J \subset K \subset I)$$

of group homomorphisms such that for all $J \subset K \subset L$, $\alpha_{J,K}\alpha_{K,L} = \alpha_{J,L}$.

There is an obvious notion of morphism of amalgams. A *completion* $\beta : A \to G$ for A is a family $\beta = (\beta_J : P_J \to G)$ of group homomorphisms such that $G = \langle P_J\beta_J : J \subset I\rangle$ and for all $J \subset K \subset I$ the obvious diagram commutes:

$$\begin{array}{ccc} P_J & \xrightarrow{\alpha_{J,K}} & P_K \\ & {\scriptstyle\beta_J}\searrow \quad \nearrow{\scriptstyle\beta_K} & \\ & G & \end{array}$$

The completion $\beta : A \to G$ is said to be *faithful* if each β_J is an injection.

The free amalgamated product $G(A)$ of A supplies a universal completion $\iota : A \to G(A)$, and if A possesses a faithful completion then the universal completion is faithful. Of course isomorphic amalgams have isomorphic universal completions.

Let $\mathcal{U} = (G, H, \Delta, \Delta_H)$ be a uniqueness system and (x, y) an edge in Δ_H. To avoid the trivial case we assume $x \neq y$. The *amalgam of* $\mathcal{U}$ is the rank

3 amalgam $A(\mathcal{U}) = (\alpha_{J,K} : P_J \to P_K)$ defined by $P_{12} = H$, $P_{23} = G_x$, $P_{13} = G(\{x,y\})$, $P_1 = H(\{x,y\})$, $P_2 = H_x$, $P_3 = G_{x,y}$, and $P_\emptyset = H_{x,y}$, with all maps $\alpha_{J,K}$ inclusions.

Observe the inclusion map $\beta : A(\mathcal{U}) \to G$ is a faithful completion of the amalgam $A(\mathcal{U})$. Let $G(A(\mathcal{U})), \iota$ be the universal completion of $A(\mathcal{U})$. Write $\tilde{G}$ for $G(A(\mathcal{U}))$, $\tilde{H}$ for $H\iota$, $\tilde{G}_{\tilde{x}}$ for $G_x\iota$, *etc.* Let $\tilde{\Delta}$ be the collinearity graph of the rank 3 coset geometry $\tilde{\Gamma}$ of $\tilde{G}$ on the image of the amalgam under ι. Then $\tilde{G}_{\tilde{x}}$ is indeed the stabilizer of some $\tilde{x} \in \tilde{\Delta}$. Let $\tilde{\Delta}_{\tilde{H}}$ be the collinearity graph of the residue of $\tilde{H}$ in $\tilde{\Gamma}$ and $\tilde{\mathcal{U}} = (\tilde{G}, \tilde{H}, \tilde{\Delta}, \tilde{\Delta}_{\tilde{H}})$. Then

(3.1) *$\tilde{\mathcal{U}}$ is a uniqueness system equivalent to $\mathcal{U}$, there exists a morphism $d : \tilde{\mathcal{U}} \to \mathcal{U}$ of uniqueness systems, and if Δ_H is a base for Δ then $\tilde{\mathcal{U}} \cong \mathcal{U}$.*

(3.2) *If $\mathcal{U}$ and $\bar{\mathcal{U}}$ are equivalent uniqueness systems then $A(\mathcal{U}) \cong A(\bar{\mathcal{U}})$.*

Notice 3.1 and 3.2 establish Theorem 1. Namely under the hypotheses of Theorem 1, 3.2 says $A(\mathcal{U}) \cong A(\bar{\mathcal{U}})$, so that $\tilde{G}$ is also the universal completion of $A(\bar{\mathcal{U}})$. Then by 3.1, $\mathcal{U} \cong \tilde{\mathcal{U}} \cong \bar{\mathcal{U}}$, as desired.

The proofs of the remaining results are more straightforward but also more technical.

Remark. The proof just sketched shows that under the hypotheses of Theorem 1, G is the free amalgamated product of H, G_x, and $G(\{x,y\})$. This observation supplies a presentation for G.

Section 4. *p*-local geometries.

In this section we adopt the terminology of Tits in [**16**] and assume:

(Γ0) G is a flag transitive group of automorphisms of a residually connected rank 3 string geometry Γ and (x, l, π) is a flag in Γ.

Consider the following hypotheses:

(Γ1) Each pair of distinct collinear points x, y is on a unique line $x + y$.
(Γ2) If $x, y \in \Gamma_1(\pi)$ are collinear then $x + y \in \Gamma_2(\pi)$.
(Γ3) Each triangle of Δ is incident with a plane.
(Γ4) $G_{\pi,l}$ is 2-transitive on $\Gamma_1(l)$.
(Γ5) $G_{x,l} = \langle G_{x,y,l}, G_{x,l,\pi}\rangle$ for $x \neq y \in \Gamma_1(l)$.

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(4.1) *Assume* (G, Γ, x, l, π) *satisfies hypotheses* (Γi) *for* $i = 0, 4, 5$. *Define* Δ *to be the collinearity graph of* Γ, $H = G_\pi$, *and* Δ_H *the collinearity graph of the residue of* π. *Then*

(1) $\mathcal{U} = (G, H, \Delta, \Delta_H)$ *is a uniqueness system.*

(2) If hypotheses (Γi), $0 \leq i \leq 5$, *hold then each triangle of* Δ *is* G-*conjugate to a triangle of* Δ_H.

Section 7 in [**2**] contains lemmas which make it possible to verify that suitable truncations of many of the p-local geometries of the sporadic groups (*cf.* [**1**]) satisfy hypotheses (Γi), $0 \leq i \leq 5$. Combining this machinery with the results in section 3 and checking the various hypotheses are satisfied, we get the following theorem whose proof is not yet written up in preprint form.

(4.2) *Assume* $\bar{G}$ *is a sporadic group with* p-*local geometry* $\bar{\Gamma}$, *where either*

(a) $p = 3$ *and* $\bar{G}$ *is* Co_1, Sz, Mc, *or* Ly, *or*

(b) $p = 2$ *and* $\bar{G}$ *is* M_{24} *or* J_4.

Assume further that (G, Γ, x, l, π) *satisfies Hypothesis* $(\Gamma 0)$ *and* $\alpha : G_x \to \bar{G}_{\bar{x}}$ *and* $\zeta : G_\pi \to \bar{G}_{\bar{\pi}}$ *are isomorphisms with* $G_{x,l}\alpha = \bar{G}_{\bar{x},\bar{l}}$, $G_{\pi,l}\zeta = \bar{G}_{\bar{\pi},\bar{l}}$, *and* $G_{x,\pi}\alpha = \bar{G}_{\bar{x},\bar{\pi}}$. *Then* $G \cong \bar{G}$ *if* Δ *and* $\bar{\Delta}$ *are triangulable.*

The p-local geometries for the sporadic groups are discussed in [**1**]. Other choices of (G, p) are possible, but the choices in (4.2) are particularly nice for a number of reasons. For example the pairs $(Sz, 2)$ and $(Ly, 5)$ are also possibilities. But in these cases the universal covering of the p-local geometry Γ is an affine building, and hence infinite. As these geometries satisfy $(\Gamma 1)$ and $(\Gamma 3)$, coverings of Γ induce coverings of the collinearity graph Δ, so Δ is not simply connected. This does not necessarily cause big problems since one might show that Δ is n-generated for some $n > 3$, but it is at least an inconvenience. In the case of $(Ly, 5)$, one of the object stabilizers is not local, so extra effort must be expended to prove the existence of this stabilizer.

To complete the treatment of the six sporadic groups listed in 4.2 requires work both at the beginning and end of the problem. To begin one must settle on a group theoretic hypothesis $\mathcal{H}$ with which to characterize G. In each of the six cases of 4.2, the optimal choice for $\mathcal{H}$ is presumably the centralizer of a 2-central involution. Thus except for Mc and Ly, this involves a large extraspecial subgroup. The next step is to prove the existence of the p-local geometry and establish the isomorphisms of 4.2. At this point 4.2 reduces the problem to a check that the collinearity graph Δ is triangulable. Because of the novelty of the approach, this last step is at present the most difficult, even though the graph Δ is the most well-behaved of those associated to G.

However there is no reason to believe that techniques can't be developed to make the check of triangulability of suitable graphs easy.

In the next section we go into some of these matters in more depth.

Section 5. Speculation.

We close with some speculation on how best to establish the uniqueness of each of the sporadic groups. But first some general discussion of the factors which need to be considered.

To begin, given a sporadic group $\bar{G}$, we need to settle on a group theoretic hypothesis $\mathcal{H} = \mathcal{H}(\bar{G})$ with which to characterize $\bar{G}$. Probably $\mathcal{H}$ will always be the general structure of the centralizer of an involution. If $\bar{G}$ possesses an involution $\bar{z}$ such that $O_2(C_{\bar{G}}(\bar{z}))$ is a large extraspecial subgroup then $C_{\bar{G}}(\bar{z})$ is probably the best choice.

The next step is to decide upon a means for establishing the uniqueness of groups G satisfying Hypothesis $\mathcal{H}$. Our predisposition is to use a graph theoretic or geometric approach. Thus in essence we must settle on a large maximal subgroup G_x of G and a self paired orbital of G on G/G_x defining a graph Δ admitting the action of G as a group of automorphisms with G_x the stabilizer of some vertex $x \in \Delta$. If possible, Δ is the collinearity graph of some geometry Γ preserved by G.

There are several things to think about in choosing Δ. First Δ will probably be easiest to work with if it is highly symmetric; that is Δ should be of small diameter and G should be of small rank on Δ. For example in using Theorem 1 this will presumably make it easier to prove Δ is triangulable.

Next it would be best if it is easy to prove the existence of G_x starting from Hypothesis $\mathcal{H}$. If G_x is a local subgroup of G then the existence of G_x is usually easy. But often the nicest subgroups are not local and hence are not easy to construct. For example it is probably easier to work with the 2-local geometry for M_{24} than to try to prove the existence of an M_{23}-subgroup starting from the structure of a 2-central involution. On the other hand the only nice graph associated to G may be on the cosets of a nonlocal subgroup, and hence the construction of this subgroup may be necessary.

Of course eventually we would like to prove the existence of all large maximal subgroups of G, so why not put in this effort at the start? There are a number of reasons to avoid such an approach. In the M_{24} example it is fairly easy to construct *some* M_{24} with an M_{23}-subgroup, so if one can prove uniqueness of M_{24} by any means then the existence of the M_{23}-subgroup follows painlessly. Also it is worth making the treatment of the uniqueness of the sporadic groups as simple and self-contained as possible. Modularization of a very large and complex undertaking is always desirable.

Finally in dealing with 26 sporadic groups, it is an advantage to introduce as much uniformity as possible and to reduce case analysis to a minimum.

Thus for example as the majority of the sporadic groups possess a large extraspecial subgroup, it is probably appropriate to emphasize such subgroups.

Now some ideas about the optimal approach to the uniqueness question for each of the sporadic groups. We have already suggested in section 4 that the groups M_{24}, J_4, Mc, Sz, Ly, and Co_1 are perhaps best viewed as acting on an appropriate p-local geometry. In [**3**], we completely implemented this approach for J_4 as a test case. Segev discusses that work elsewhere in these Proceedings.

We believe He is best represented on a rank 3 geometry with two $S_6/\mathbf{Z}_3/E_{64}$ stabilizers and one $S_3/L_3(4)/E_4$ stabilizer, with the collinearity graph defined on the cosets of the final stabilizer. The notion of "uniqueness system" discussed in section 3 is not quite applicable here, but a very slight generalization can be applied.

The groups F_1, F_2, and F_5 are perhaps best represented on a commuting graph on a class of non-2-central involutions. This approach has been implemented by Griess, Meierfrankenfeld, and Segev in [**5**] for the Monster, and by Segev in [**10**] and [**11**] for F_2 and F_5. The results in [**2**] described in section 4 grew out of attempts to understand the approach in these papers, and those results can be used to greatly simplify the treatments in [**5**], [**10**], and [**11**]. We have written out such a simplification for the Monster in section 8 of [**2**] as another test case.

The three Fischer groups are probably best viewed as 3-transposition groups. Given the 3-transposition theory, the proof of the uniqueness of the Fischer groups is elegant and easy. Moreover it is worth developing this theory for a variety of other reasons.

The four small Mathieu groups could be handled by any of a number of means. There is great room for ingenuity here. For example several years ago an undergraduate at Caltech named Laura Anderson (now a graduate student at MIT) produced a simple proof of the uniqueness of M_{12}.

Similarly it is not clear which approach is optimal for the three small Janko groups. Janko proves J_1 is unique as a 7-dimensional linear group over $GF(7)$ in [**7**]. One could also consider the action on an $L_2(11)$-subgroup. Hall and Wales show J_2 is unique in [**6**] via constructing a $U_3(3)$-subgroup. Another possibility is to consider the commuting graph on 3-central subgroups of order 3. Considering the small size of J_3, We see no attractive way to prove its uniqueness. Frohardt has established uniqueness in [**4**] via a trilinear form in characteristic 0 but his proof requires some sweat.

One could approach the uniqueness of Co_2 via either its 2 or 3-local geometry. The latter geometry is nicer but it involves the construction of a $\mathbf{Z}_2/U_6(2)$-subgroup. The existing proof is due to F. Smith in [**13**], which takes this approach.

Similarly Co_3 could be approached via its 2 or 5-local geometry, with the