A SURVEY OF RECENT RESULTS ON PROJECTIVE REPRESENTATIONS OF THE SYMMETRIC GROUPS

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The basic results on the projective representations of the symmetric groups were obtained by Schur in 1911, [14]. In recent years, there has been considerable interest in this area. The aim of this article is to outline some of the recent developments. The articles by Stembridge [15] and Józefiak [7] and the forthcoming book [5], provide introductions to the subject.

1. Preliminaries

Definition. A *projective representation* of a group G of degree d over a field K is a map $P : G \to GL(d,K)$ such that

(a) $P(1_G) = I_d$;

and

(b) given x and y in G, there is an element $\alpha(x,y)$ in $K^{\times}$ (the multiplicative group of K) such that

$$P(x)\,P(y) = \alpha(x,y)\,P(xy).$$

When $\alpha(x,y) = 1$ for all x and y, we say that P is a *linear representation* of G.

Using the fact that the matrices $\{P(g) : g \in G\}$ are invertible, the fact that $P(1_G)$ is the identity matrix gives:

(C1) for all g in G,

$$\alpha(g,1) = 1 = \alpha(1,g).$$

Using invertibility again and also associativity of group composition and of matrix multiplication, (b) gives:

(C2) for all x, y and z in G,

$$\alpha(xy, z)\alpha(x,y) = \alpha(x, yz)\alpha(y,z).$$

A map $\alpha : G \times G \to K^{\times}$ satisfying (C1) and (C2) is a *2-cocycle*.

There is an equivalence relation on the set of 2-cocycles: α is equivalent to β if and only if there is a map $\delta : G \to K^{\times}$ such that for all x, y in G

$$\alpha(x,y) = \delta(x)\delta(y)\delta(xy)^{-1}\beta(x,y).$$

The equivalence classes of 2-cocycles form a group with multiplication $[\alpha\beta] = [\alpha][\beta]$ where

$$\alpha\beta(x,y) = \alpha(x,y)\beta(x,y).$$

This group is usually denoted $H^2(G,K^{\times})$. When K is algebraically closed of characteristic zero, it is known as the *Schur multiplier* and is often denoted M(G).

Under suitable assumptions of K (for example if K is algebraically closed of characteristic zero), the group G has a K-*representation group*. This is a group H with central subgroup A, isomorphic to $H^2(G, K^{\times})$, with H/A isomorphic to G such that every projective representation of G over K can be "lifted" to a linear representation of H. In these circumstances, the projective representations of G may be regarded as linear representations of its representation group.

2. The symmetric groups

For the symmetric group S(n), Schur showed in [14] that the multiplier is cyclic of order 2 when $n \geq 4$. Since complex representation groups always exist, there is a group $\tilde{S}(n)$ of order 2n! which is a "double cover" of S(n). Schur provided an explicit matrix representation of $\tilde{S}(n)$ (the *basic representation*) as follows. Consider the complex matrices

$$A = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then

$$A^2 = B^2 = -I; \qquad C^2 = I; \qquad AB = -BA; \qquad AC = -CA; \qquad BC = -CB.$$

Now let m be the integer part of (n-1)/2, and define the following $2^m \times 2^m$ matrices:

$$M_{2m+1} = iC^{\otimes m}$$

and for $1 \le k \le m$,

$$M_{2k-1} = C^{\otimes(m-k)} \otimes A \otimes I^{\otimes(k-1)};$$

$$M_{2k} = C^{\otimes(m-k)} \otimes B \otimes I^{\otimes(k-1)}.$$

Thus $M_j^2 = -I$ and $M_jM_k = -M_kM_j$ $(j \neq k)$. Finally, for $1 \le k \le n-1$, let

$$T_k = -\left(\frac{k-1}{2k}\right)^{1/2} M_{k-1} + \left(\frac{k+1}{2k}\right)^{1/2} M_k.$$

It may be checked that these matrices satisfy the relations

$$T_k^2 = -I \qquad (1 \le k \le n-1);$$

$$(T_kT_{k+1})^3 = -I \qquad (1 \le k \le n-2);$$

$$T_jT_k = -T_kT_j \qquad (1 \le j,\ k \le n-1, \quad |j-k| > 1),$$

and so generate the required representation P_n of $\tilde{S}(n)$ with degree 2^m. This representation is irreducible and *negative* in the sense that the central involution of $\tilde{S}(n)$ is represented as the negative of the identity matrix. Replacing T_k by $-T_k$ gives another representation P'_n which is equivalent to P_n if and only if n is odd.

Schur [14], showed that the irreducible negative representations of $\tilde{S}(n)$ can be indexed by *strict partitions* of n, that is partitions of n with no repeated parts. This indexing is not, however, bijective. If λ has ℓ non-zero parts, there are two irreducible negative representations corresponding to λ if $n-\ell$ is odd. The two irreducibles in this case are *associated*, in that one is obtained from the other by forming the tensor product with the sign representation. There is a unique irreducible associated with λ when $n-\ell$ is even. The basic irreducible negative representation P_n corresponds to the partition (n).

3. The Q-functions

We next give a combinatorial definition of the Q-functions. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a strict partition of n with ℓ non-zero parts. *A shifted Young diagram of shape* λ is a diagram with ℓ rows and λ_i nodes in each row with the first node in row i+1 being under the second node in row i.

In order to define the Q-functions, we let **P** denote the ordered alphabet $\{1' < 1 < 2' < 2 < 3' \ldots\}$. The letters $1', 2', 3', \ldots$ are *marked* (and 1, 2, 3, ... *unmarked*). The notation $|a|$ is used for the unmarked version of any element a of **P**.

Definition. A *shifted Young tableau of shape* λ is an assignment of elements of **P** to the nodes of a shifted Young diagram of shape λ such that

(i) the entries are weakly increasing along rows and down columns;

(ii) there is at most one occurrence of any given unmarked letter in any given column,

and

(iii) there is at most one occurrence of any given marked letter in each row.

Example. A shifted Young tableau associated with the partition $\lambda = (7, 5, 3, 2, 1)$ is

1	1	2′	2	2	4	5
	2′	2	3	4	6	
		3	5	6		
			5′	7′		
				7		

The *content* of a shifted Young tableau T of shape λ is the sequence $\gamma = (\gamma_1, \gamma_2, \ldots)$ where γ_i is the number of entries of T equal to $|i|$. In the above example $\gamma = (2, 5, 2, 2, 3, 2, 2)$. For any set $\{x_1, x_2, \ldots\}$ of indeterminates, let

$$x^T = x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \ldots .$$

Definition. Let λ be a strict partition. Define the Q-function $Q_\lambda = \sum_T x^T$, where the sum is over all shifted Young tableaux of shape λ.

The Q-functions are in fact symmetric functions, as was clear from the original definition of Schur. They play a role for projective representations analogous to that played in the linear representation theory by the Schur functions. The Q-functions can also be regarded as a special case of the Hall-Littlewood polynomials. The details of these alternative descriptions and their equivalence may be found in [5].

4. Recent advances

(a) In 1988, Nazarov [11] announced a method to construct the matrices representing the generators of $\tilde{S}(n)$ for the irreducible negative representations associated with the strict partition λ. His method is to define matrices of the appropriate size, and to check that they do indeed satisfy the relations for the group $\tilde{S}(n)$. The details of this construction may be found in [12].

(b) Recursive formulae to calculate the value of a given irreducible negative character at a specific conjugacy class of $\tilde{S}(n)$ in terms of the values of the characters of $\tilde{S}(k)$, for $k < n$ have been obtained by Morris [9] and Morris-Olsson [10]. This is analogous to the Murnaghan-Nakayama formula for linear characters of $S(n)$.

(c) It is possible to write the product of Q-functions $Q_\lambda Q_\mu$ in the form $\Sigma f^{\nu}_{\lambda\mu} Q_\nu$, where the coefficients can be proved to be non-negative integers. In fact Stembridge [15] has given a combinatorial way to calculate the coefficients $f^{\nu}_{\lambda\mu}$, similar to the Littlewood-Richardson rule for linear representations. The coefficients are somewhat complicated to describe, and the proof of Stembridge's result makes use of results of Worley [16].

(d) There is an algebra associated with the negative representations of the groups $\tilde{S}(n)$, analogous to the well-known graded algebra of Grothendieck groups of linear representations of $S(n)$. This is obtained as follows. For $n \geq 4$, let T^0_n be the Grothendieck group of isomorphism classes of finite-dimensional negative representations of $\tilde{S}(n)$, and let T^1_n be the Grothendieck group of isomorphism classes of finite-dimensional negative representations of $\tilde{A}(n)$. Let T^*_n be the $\mathbf{Z}/2$-graded group $T^0_n \oplus T^1_n$. Let L be the ring $\mathbf{Z}[\lambda]/(\lambda^3 - 2\lambda)$, which is $Z/2$-graded by letting $L^0 = \mathbf{Z} \oplus \rho\mathbf{Z}$, (where $\rho = \lambda^2 - 1$) and $L^1 = \lambda Z$. Then, with appropriate definitions of T^*_n for values of $n < 4$, $\mathfrak{J} = \oplus_{n\geq 1} T^*_n$ is a graded L-module. In fact $\mathfrak{J}$ is an algebra (induction product) and coalgebra (restriction) so that $\mathfrak{J}$ is a Hopf algebra. This algebra approach to the subject is investigated in Hoffman and Humphreys, [3] and [4].

(e) The representation theory over modular fields is the subject of investigations by several authors. When p = 2, the modular representations of $\tilde{S}(n)$ coincide with those for S(n). The irreducible negative complex characters then provide a useful class of Brauer characters for S(n) (see Benson [1]).

For odd primes p, the assignment of irreducible complex characters to p-blocks has been given by Humphreys [6] in answer to a conjecture of Morris. Alternative proofs of this result have been given by Cabanes [2] and Olsson [13]. Michler and Olsson [8] have recently shown that the Alperin-McKay conjecture holds in $\tilde{S}(n)$ for odd primes.

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SOME APPLICATIONS OF GRADED DIAGRAMS IN COMBINATORIAL GROUP THEORY

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This paper is based on four lectures given by the second author at the conference "Groups - St Andrews 1989", which in turn were based on his book [1] and on results obtained recently by the authors and others at the higher algebra seminar of Moscow State University.

1. Diagrams over groups

1.1 van Kampen's lemma. Let G be a group generated by a set $\{a_1, a_2, \ldots\}$ of generators and a set of defining relations $\{r_1 = 1, r_2 = 1, \ldots\}$, where $r_1, r_2 \ldots$ are words in the group alphabet

$$\mathfrak{A} = \{a_1^{\pm 1}, a_2^{\pm 1}, \ldots\}.$$

Of course, any group G can be described by means of such a presentation

$$G = \langle a_1, a_2, \ldots \mid r_1 = 1, r_2 = 1, \ldots \rangle.$$

Van Kampen [2] discovered a very simple visual demonstration of the deducibility of consequences of defining relations. For example, the equality

$$a^2b\,ab^{-1} = 1$$

follows from relations

$$a^3 = 1 \text{ and } aba^{-1}b^{-1} = 1$$

and the deducibility is pictured in Fig. 1. We read one of the defining words by going around any region and we read the consequence by going along the boundary of the map (following the inverse edge gives the inverse letter).

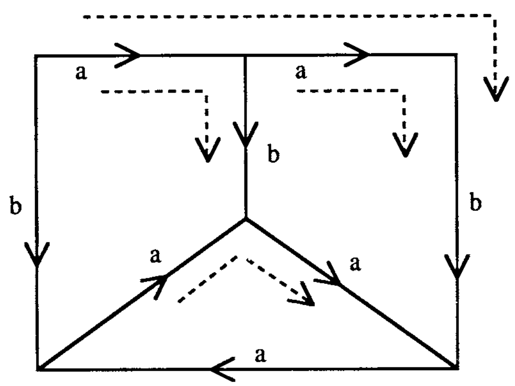

Fig. 1

Let us give some definitions.

A *map* is a finite connected, simply connected planar two-complex. Let

$$\mathfrak{A} = \{a_1^{\pm 1}, a_2^{\pm 1}, \ldots\}$$

be a (group) *alphabet*. *A diagram* M over $\mathfrak{A}$ is a map which is equipped with a labelling function ϕ from the set of oriented edges (one-cells) of M into $\mathfrak{A}$ with the property that for any edge e of M

$$\phi(e^{-1}) = \phi(e)^{-1}.$$

Let G be a group defined by the presentation

$$G = < \mathfrak{A} \mid \mathfrak{R} >, \qquad (1.1)$$

where $\mathfrak{R}$ is a set of relators of G.

A diagram M over G (more accurately, over the presentation (1.1)) is a diagram M over $\mathfrak{A}$ such that if p is a boundary cycle for a region (= two-cell) of M, then the word $\phi(p)$ is an element of $\mathfrak{R}$.

Of course, the word $\phi(p)$ depends on a choice of an initial vertex on the boundary of the cell and on the choice of direction (clockwise or counterclockwise). However, this causes no trouble, because we can assume that the set of defining words $\mathfrak{R}$ is *symmetrized*:

(a) $r \in \mathfrak{R} \Rightarrow r^{-1} \in \mathfrak{R}$;

(b) $r = uv \in \mathfrak{R} \Rightarrow$ cyclic conjugate $vu \in \mathfrak{R}$;

(c) $r \in \mathfrak{R} \Rightarrow r$ is reduced, i.e. without subwords of the type aa^{-1}, where $a \in \mathfrak{A}$.

The statement of van Kampen's lemma is almost obvious [3].

A relation $w = 1$ *holds in the group* G *presented by* (1.1) *if and only if there exists a diagram* M *over* G *with boundary path* $p = e_1 \ldots e_m \in M$ *such that* $\phi(e_1) \ldots \phi(e_m) \equiv w$ (where $\equiv$ denotes graphical equality).

Moreover, one may assume that M is a *reduced diagram*. What does this mean? Let us suppose there exists a pair of regions S_1, S_2 with a common edge e such that for boundary paths $p_1 = eq_1$ and $p_2 = eq_2$, beginning with e, the equation $w_1 = w_2$ holds, where the words w_1, w_2 are the labels of p_1 and p_2. Then we can cut out two regions and sew up the hole (see Fig. 2). And so we can assume that the diagram M in van Kampen's lemma does not contain such a pair of contractible regions. In this case, we say that M is a reduced diagram.

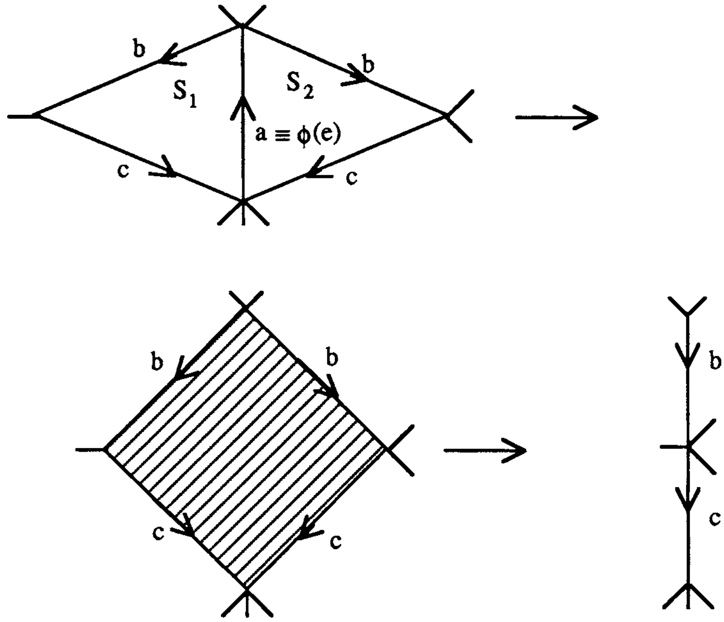

Fig. 2