A course in mathematics for students of physics: 1

A COURSE IN

mathematics

FOR STUDENTS OF
PHYSICS: 1

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CONTENTS OF VOLUME 1

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Paul Bamberg and Shlomo Sternberg
Frontmatter
More information

CONTENTS OF VOLUME 2

PREFACE

This book, with apologies for the pretentious title, represents the text of a course we have been teaching at Harvard for the past eight years. The course is aimed at students with an interest in physics who have a good grounding in one-variable calculus. Some prior acquaintance with linear algebra is helpful but not necessary. Most of the students simultaneously take an intensive course in physics and so are able to integrate the material learned here with their physics education. This also is helpful but not necessary. The main topics of the course are the theory and physical application of linear algebra, and of the calculus of several variables, particularly the exterior calculus. Our pedagogical approach follows the 'spiral method' wherein we cover the same topic several times at increasing levels of sophistication and range of application, rather than the 'rectilinear approach' of strict logical order. There are, we hope, no vicious circles of logical error, but we will frequently develop a special case of a subject, and then return to it for a more general definition and setting only after a broader perspective can be achieved through the introduction of related topics. This makes some demands of patience and faith on the part of the student. But we hope that, at the end, the student is rewarded by a deeper intuitive understanding of the subject as a whole.

Here is an outline of the contents of the book in some detail. The goal of the first four chapters is to develop a familiarity with the algebra and analysis of square matrices. Thus, by the end of these chapters, the student should be thinking of a matrix as an object in its own right, and not as a square array of numbers. We deal in these chapters almost exclusively with 2×2 matrices, where the most complicated of the computations can be reduced to solving quadratic equations. But we always formulate the results with the higher-dimensional case in mind. We begin Chapter 1 by explaining the relation between the multiplication law of 2×2 matrices and the geometry of straight lines in the plane. We develop the algebra of 2×2 matrices and discuss the determinant and its relation to area and orientation. We define the notion of an abstract vector space, in general, and explain the concepts of basis and change of basis for one- and two-dimensional vector spaces.

In Chapter 2 we discuss conformal linear geometry in the plane, that is, the geometry of lines and angles, and its relation to certain kinds of 2×2 matrices. We also discuss the notion of eigenvalues and eigenvectors, so important in quantum mechanics. We use these notions to give an algorithm for computing the powers of a matrix. As an application we study the basic properties of Markov chains.

The principal goal of Chapter 3 is to explain that a system of homogeneous linear differential equations with constant coefficients can be written as $d\mathbf{u}/dt = A\mathbf{u}$ where A is a matrix and $\mathbf{u}$ is a vector, and that the solution can be written as $e^{At}\mathbf{u}_0$ where $\mathbf{u}_0$ gives the initial conditions. This of course requires us to explain what is meant by the exponential of a matrix. We also describe the qualitative behavior of solutions and the inhomogeneous case, including a discussion of resonance.

Chapter 4 is devoted to the study of scalar products and quadratic forms. It is rich in physical applications, including a discussion of normal modes and a detailed treatment of special relativity.

Chapters 5 and 6 present the basic facts of the differential calculus. In Chapter 5 we define the differential of a map from one vector space to another, and discuss its basic properties, in particular the chain rule. We give some physical applications such as Kepler motion and the Born approximation. We define the concepts of directional and partial derivatives, and linear differential forms.

In Chapter 6 we continue the study of the differential calculus. We present the vector versions of the mean-value theorem, of Taylor's formula and of the inverse function theorem. We discuss critical point behavior and Lagrange multipliers.

Chapters 7 and 8 are meant as a first introduction to the integral calculus. Chapter 7 is devoted to the study of linear differential forms and their line integrals. Particular attention is paid to the behavior under change of variables. Other one-dimensional integrals such as arc length are also discussed.

Chapter 8 is devoted to the study of exterior two-forms and their corresponding two-dimensional integrals. The exterior derivative is introduced and invariance under pullback is stressed. The two-dimensional version of Stokes' theorem, i.e. Green's theorem, is proved. Surface integrals in three-space are studied.

Chapter 9 presents an example of how the results of the first eight chapters can be applied to a physical theory – optics. It is all in the nature of applications, and can be omitted without any effect on the understanding of what follows.

In Chapter 10 we go back and prove the basic facts about finite-dimensional vector spaces and their linear transformations. The treatment here is a straightforward generalization, in the main, of the results obtained in the first four chapters in the two-dimensional case. The one new algorithm is that of row reduction. Two important new concepts (somewhat hard to get used to at first) are introduced: those of the dual space and the quotient space. These concepts will prove crucial in what follows.

Chapter 11 is devoted to proving the central facts about determinants of $n \times n$

matrices. The subject is developed axiomatically, and the basic computational algorithms are presented.

Chapters 12–14 are meant as a gentle introduction to the mathematics of shape, that is, algebraic topology. In Chapter 12 we begin the study of electrical networks. This involves two aspects. One is the study of the 'wiring' of the network, that is, how the various branches are interconnected. In mathematical language this is known as the topology of one-dimensional complexes. The other is the study of how the network as a whole responds when we know the behavior of the individual branches, in particular, power and energy response. We give some applications to physically interesting networks.

In Chapter 13 we continue the study of electrical networks. We examine the boundary-value problems associated with capacitive networks and use these methods to solve some classical problems in electrostatics involving conductors.

In Chapter 14 we give a sketch of how the one-dimensional results of Chapters 12 and 13 generalize to higher dimensions.

Chapters 15–18 develop the exterior differential calculus as a continuous version of the discrete theory of complexes. In Chapter 15 the basic facts of the exterior calculus are presented: exterior algebra, k-forms, pullback, exterior derivative and Stokes' theorem.

Chapter 16 is devoted to electrostatics. We suggest that the dielectric properties of the vacuum give the continuous analog of the capacitance of a network, and that these dielectric properties are what determine Euclidean geometry in three-dimensional space. The basic facts of potential theory are presented.

Chapter 17 continues the study of the exterior differential calculus. The main topics are vector fields and flows, interior products and Lie derivatives. These are applied to magnetostatics.

Chapter 18 concludes the study of the exterior calculus with an in-depth discussion of the star operator in a general context.

Chapter 19 can be thought of as the culmination of the course. It applies the results of the preceding chapters to the study of Maxwell's equations and the associated wave equations.

Chapters 20 and 21 are essentially independent of Chapters 9–19 and can be read independently of them. They are not usually included in our one-year course. But Chapters 1–9, 20 and 21 would form a self-contained unit for a shorter course.

The material in Chapter 20 is a relatively standard treatment of the theory of functions of a complex variable, suitable for students at the level of this book.

Chapter 21 discusses some of the more elementary aspects of asymptotics.

Chapter 22 shows how the exterior calculus can be used in classical thermodynamics, following the ideas of Born and Carathéodory.

The book is divided into two volumes, with Chapters 1–11 in volume 1.

Most of the mathematics and all of the physics presented in this book were developed by the first decade of the twentieth century. The material is thus at least seventy-five years old. Yet much of the material is not yet standard in the elementary courses (although most of it with the possible exception of network theory must be learned for a grasp of modern physics, and is studied at some stage of the physicist's career). The reasons are largely historical. It was apparent to Hamilton that the real and complex numbers were insufficient for the deeper study of geometrical analysis, that one wants to treat the number pairs or triplets of the Cartesian geometry in two and three dimensions as objects in their own right with their own algebraic properties. To this end he developed the algebra of quaternions, a theory which had a good deal of popularity in England in the middle of the nineteenth century. Quaternions had several drawbacks: they more naturally pertained to four, rather than to three dimensions – the geometry of three dimensions appeared as a piece of a larger theory rather than having a natural existence of its own; also, they have *too much* algebraic structure, the relation between quaternion multiplication, for example, and geometric constructions in three dimensions being somewhat complicated. (The first of these objections would, of course be regarded far less seriously today. But it would be replaced by an objection to a theory that is *limited* to four dimensions.) Eventually, the three-dimensional *vector algebra* with its scalar and vector products was distilled from the theory of quaternions. It was conjoined with the necessary differential operations, and give rise to the *vector analysis* as finally developed by Gibbs and promulgated by him in a famous and very influential text.

So vector analysis, with its grad, div, curl etc. became the standard language in which the geometric laws of physics were taught. Now while vector analysis is well suited to the geometry of three-dimensional Euclidean space, it has a number of serious drawbacks. First, and least serious, is that the essential unity of the subject is obscured. Thus the fundamental theorem of the calculus, Green's theorem, Gauss' theorem and Stokes' theorem are all aspects of the same theorem (now called Stokes' theorem). But this is not at all clear in the vector analysis treatment. More serious is that the fundamental operators involve the Euclidean structure (for example grad and div) or the three-dimensional structure and orientation as well (for example curl). Thus the theory is wedded to a three-dimensional orientated Euclidean space. A related problem is that the operators do not behave nicely under general changes of coordinates – their expression in non-rectangular coordinates being unwieldy. Already Poincaré, in his fundamental scientific and philosophical writings which led to the theory of relativity, stressed the need to distinguish between those laws of geometry and physics which are 'topological', i.e. depend only on the differential structure of space and so are invariant under smooth deformations, and those which depend on more geometrical structure such as the notion of distance. One of the major impacts of the theory of relativity on mathematics was to encourage the study of higher-dimensional spaces, a study which had existed in the previous mathematical literature, but was not regarded as central to the study of geometry. Another was to emphasize general coordinate changes. The vector analysis was not up to these two tasks and so was supplemented in the more advanced literature by *tensor analysis*. But tensor analysis with its jumble of indices has a number of serious drawbacks, the most serious of which being that it is extraordinarily difficult to tell which operations have any geometric significance and which are artifacts of the coordinate system. Thus, while it is reasonably well-suited for computation, it is hard to assess exactly what it is that one is computing. The whole purpose of the development initiated by Hamilton – to have a calculus whose objects have a perceived geometrical significance – was vitiated. In order to make the theory work one had to introduce a relatively sophisticated geometrical construct, such as an affine connection. Even with such constructs the geometric meanings of the operations are obscure. In fact tensor analysis never displaced the intuitively clear vector analysis from the elementary curriculum.

It is generally accepted in the mathematics community, and gradually being accepted in the physics community, that the most suitable framework for geometrical analysis is the exterior differential calculus of Grassmann and Cartan. This calculus has the advantage that its computational rules are simple and concise, that its objects have a transparent geometrical significance, that it works in all

Maxwell's equations in the course of history
The constants c, μ_0, and ε_0 are set to 1.

The homogeneous equation	**The inhomogeneous equation**
Earliest form	
$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=0$	$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=\rho$
$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-\dot{B}_x$	$\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}=j_x+\dot{E}_x$
$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=-\dot{B}_y$	$\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}=j_y+\dot{E}_y$
$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-\dot{B}_z$	$\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}=j_z+\dot{E}_z$
At the end of the last century	
$\nabla\cdot\mathbf{B}=0$ $\nabla\times\mathbf{E}=-\dot{\mathbf{B}}$	$\nabla\cdot\mathbf{E}=\rho$ $\nabla\times\mathbf{B}=\mathbf{j}+\dot{\mathbf{E}}$
At the beginning of this century	
${}^*F^{\beta\alpha}{}_{,\alpha}=0$	$F^{\beta\alpha}{}_{,\alpha}=j^\beta$
Mid-twentieth-century	
$\mathrm{d}F=0$	$\delta F=J$

dimensions, that it behaves well under maps and changes of coordinates, that it has an essential unity to its principal theorems and that it clearly distinguishes between the 'topological' and 'metrical' properties. The geometrical laws of physics take on a simple and elegant form in terms of the exterior calculus. To emphasize this point, it might be useful to reproduce the above table, taken from Thirring's *Course on Mathematical Physics.*

Hermann Grassmann (1809–77) published his *Ausdehnungslehre* in 1844. It was not appreciated by the mathematical community and was dismissed by the leading German mathematicians of his time. In fact, Grassmann was never able to get a university position in mathematics. He remained a high-school teacher throughout his career. (Nevertheless, he seemed to have a happy and productive life. He raised a large family and was recognized as an expert on Sanskrit literature.) Towards the end of his life he tried again, with another edition of his *Ausdehnungslehre*, but this fared no better than the first. Only one or two mathematicians of his time, such as Möbius, appreciated his work. Nevertheless, the *Ausdehnungslehre* (or calculus of extension) contains for the first time many of the notions central to modern mathematics and most of the algebraic structures used in this book. Thus vector spaces, exterior algebra, exterior and interior products and a form of the generalized Stokes' theorem all make their appearance.

Elie Cartan (1869–1951) is now universally recognized as the leading geometer of our century. His early work, of such overwhelming importance for modern mathematics, on Lie groups and on systems of partial differential equations was done in relative obscurity. But, by the 1920s, his work became known to the broad mathematical community, due, in part, to the writings of Hermann Weyl who presented novel expositions of his work at a time when the theory of Lie groups began to play a central role in mathematics and in physics. Cartan's work on the theory of principal bundles and connections is now basic to the theory of elementary particles (where it goes under the generic name of 'gauge theories'). In 1922 Cartan published his book *Leçons sur les invariants intégraux* in which he showed how the exterior differential calculus, which he had invented, was a flexible tool, not only for geometry but also for the variational calculus and a wide variety of physical applications. It has taken a while, but, as we have mentioned above, it is now recognized by mathematicians and physicists that this calculus is the appropriate vehicle for the formulation of the geometrical laws of physics. Accordingly, we feel that it should displace the 'vector calculus' in the elementary curriculum and have proceeded accordingly.

Some explanation is in order for the time and effort devoted to the theory of electrical networks, a subject not usually considered as part of the elementary curriculum. First of all there is a purely pedagogical justification. The subject always goes over well with the students. It provides a down-to-earth illustration of such concepts as dual space and quotient space, concepts which frequently seem overly abstract and not readily accepted by the student. Also, in the discrete, algebraic setting of network theory, Stokes' theorem appears as essentially a definition, and a natural one at that. This serves to motivate the d operator and Stokes' theorem in the setting of the exterior calculus. There are deeper, more philosophical reasons for our decision to emphasize network theory. It has been recognized for about a century that the forces that hold macroscopic bodies together are essentially electrical in character. Thus (in the approximation where the notion of rigid body and Euclidean geometry makes sense, that is, in the non-relativistic realm) the concept of a rigid body, and hence of Euclidean geometry, derives from electrostatics. The frontiers of physics, both in the very small (the study of elementary particles) and the very large (the study of cosmology) have already begun to reopen fundamental questions as to the geometry of space and time. We thought it wise to bring some of the issues relating geometry to physics before the student even at this early stage of the curriculum. The advent of the computer, and also some of the recent theories of physics will, no doubt, call into question the discrete versus the continuous character of space and time (an issue raised by Riemann in his dissertation on the foundations of geometry). It is to be hoped that our discussion may be of some use to those who will have to deal with this problem in the future.

Of course, we have had to omit several important topics due to the limitation of a one-year course. We do not discuss infinite-dimensional vector spaces, in particular Hilbert spaces, nor do we define or study abstract differentiable manifolds and their properties. It has been our experience that these topics make too heavy a demand on the sophistication of the student, and the effort involved in explaining them is best expended elsewhere. Of course, at various places in the text we have to pay the price for not having these concepts at our disposal. More serious is the omission of a serious discussion of Fourier analysis, classical mechanics and probability theory. These topics are touched upon but not presented as a coherent subject of study. Our only excuse is that a thorough study of each would probably require a semester's course, and substantive treatments from the modern viewpoint are available elsewhere. A suggested guide to further reading is given at the end of the book.

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