1

Linear transformations of the plane

In Chapter 1 we explain the relation between the multiplication law of 2×2 matrices and the geometry of straight lines in the plane. We develop the algebra of 2×2 matrices and discuss the determinant and its relation to area and orientation. We define the notion of an abstract vector space, in general, and explain the concepts of basis and change of basis for one- and two-dimensional vector spaces.

1.1. Affine planes and vector spaces

The familiar Euclidean plane of high-school plane geometry arose early in the history of mathematics because its properties are readily discovered by physical experiments with a tabletop or blackboard. Through our experience in using rulers and protractors, we are inclined to accept 'length' and 'angle' as concepts which are as fundamental as 'point' and 'line'. We frequently have occasion, though, both in pure mathematics and in its applications to physics and other disciplines, to consider planes for which straight lines are defined but in which no general notion of length is defined, or in which the usual Euclidean notion of length is not appropriate. Such a plane may be represented on a sheet of paper, but the physical distance between two points on the paper, as measured by a ruler, or the angle between two lines, as measured by a protractor, need have no significance.

An example of such a plane is the one used to describe graphically the motion of particles along a line (the x-axis). A point P or Q in this plane represents the physical concept of *event*, something which has a time and place. A line l also

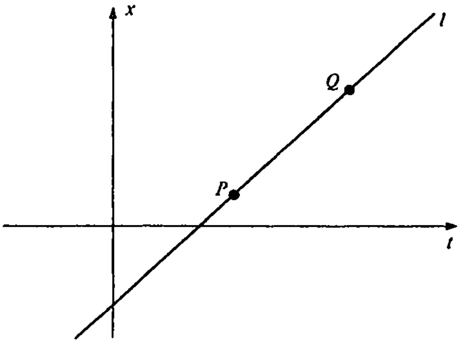

Figure 1.1

has physical significance; it corresponds to the motion of a particle which is subject to no force. We can compare the lengths of segments along the t-axis (time intervals) or along the x-axis (distances). Yet the distance between P and Q, as measured with a ruler, is devoid of physical significance. Furthermore, the origin in such a plane, where the axes cross, is of no fundamental physical significance.

The mathematical concept of *real affine plane* is the appropriate one to represent this and many other 'two-dimensional' situations. An affine plane contains *points*, which we shall represent by upper case letters P, Q, etc., and straight lines, which we shall call simply *lines* and represent by lower-case letters l, m, etc. As our model for the affine plane, we shall follow Descartes and consider the set of all pairs of real numbers as our plane. A typical point is then an ordered pair of real numbers denoted by $\begin{bmatrix} x \\ y \end{bmatrix}$. This plane is called $A\mathbb{R}^2$. The A stands for affine, and is to remind us that we have no preferred origin. The $\mathbb{R}$ stands for the collection of real numbers, and the superscript 2 indicates that we are considering pairs of real numbers. (When we plot the plane on paper, the usual convention is to plot x horizontally and y vertically along perpendicular axes. However, the notion of 'perpendicular' or the size of any angle is undefined for us at the moment. We could just as well plot x and y along any axes.) A line is a particular kind of set of points. We assume that you are familiar with (straight) lines from your previous studies of geometry, and, in particular, that you are acquainted with the description of lines in analytic geometry.

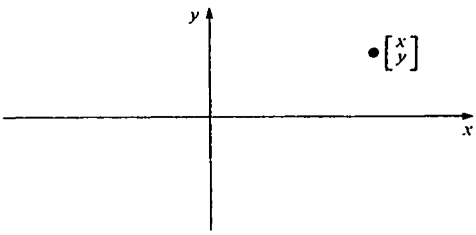

Figure 1.2

The lines of the affine plane $\mathsf{A}\mathbb{R}^2$ can be described in various ways. One way is to give an equation satisfied by the points of the line, for example

$$l = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| ax + by = c \right\}.$$

This is to be read as 'l is the set of points $\begin{bmatrix} x \\ y \end{bmatrix}$ such that the equation $ax + by = c$ is satisfied'. Here it is assumed that a and b are not both zero.

This method of characterizing a line is a little inconvenient because the parameters a, b, c which characterize the line are not unique. For example

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| ax + by = c \right\}$$

and

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| 3ax + 3by = 3c \right\}$$

are the same line. More generally the parameters ra, rb, rc, for $r \neq 0$, describe the same straight line as a, b, c.

A second method of characterizing a line in $\mathsf{A}\mathbb{R}^2$ is in terms of two points lying on the line. Given two distinct points $P_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and $P_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, we construct the line through P_1 and P_2 as the set of all points

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 + t(x_1 - x_0) \\ y_0 + t(y_1 - y_0) \end{bmatrix}$$

where the parameter t ranges over the real numbers. This description of a line is even more redundant than the previous one: we can replace our points P_1 and P_0 by any other pair of distinct points on the same line.

Another convenient way of describing a straight line (a more 'dynamic' as opposed to a 'static' way) is to give a point on the line and the 'direction vector of the line': thus the set of all points of the form

$$\left\{ \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{pmatrix} u \\ v \end{pmatrix} \middle| t \in \mathbb{R} \right\} \quad \text{where} \quad \begin{pmatrix} u \\ v \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ is a fixed vector}$$

is a line. $\Bigg($Here we think of the line as being traversed by a particle moving with 'velocity vector' $\begin{pmatrix} u \\ v \end{pmatrix}$ and situated at $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ at time zero.$\Bigg)$ Here we have used four parameters to describe the line. But we can multiply $\begin{pmatrix} u \\ v \end{pmatrix}$ by any non-zero scalar and get the same line (just traversed with different velocity) and we can displace $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ along the line, showing that we have two redundant parameters.

Of course, this ties in with our second description if

$$u = x_1 - x_0,$$
$$v = y_1 - y_1.$$

There is a fourth, familiar description of a line which is not redundant, but has the awkward feature that it does not describe absolutely all lines in the same way. If a and b are any real numbers, the set

$$l = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| y = ax + b \right\}$$

is a straight line which intersects the y-axis at the point $\begin{bmatrix} 0 \\ b \end{bmatrix}$ and which has 'slope' a; i.e., for points on the line, an increase in one unit of x implies an increase in a units of y. This set is a line, and the description is not redundant, for we have described a and b in terms of geometric properties of the line. But not all lines are of this form. We must add the lines which are parallel to the y-axis, and which have the description

$$l = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| x = d \right\}$$

From a strictly logical point of view, we should take one of the four descriptions given above as our *definition* of a straight line; for example, we should say that, by definition, a line is a subset, l, of $A\mathbb{R}^2$ such that there are three real numbers a, b, and c with a and b not both zero such that

$$l = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| ax + by = c \right\}.$$

We should then *prove* that such a subset can be given by either of the other three descriptions. We shall not go into such logical niceties here, since you have seen, or can construct, such arguments from elementary analytic geometry.

It is important to remember that an affine plane has no origin and that it makes no sense to add points of an affine plane. We attach no special significance to the point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and we resist the temptation to add points like $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ 'coordinate by coordinate'. There is, however, a closely related mathematical structure, called a two-dimensional *vector space*, in which an operation of addition *is* defined. We construct a vector space from an affine plane by associating with any pair of points the 'displacement vector' $\overrightarrow{PQ}$ whose 'tail' is at P and whose 'head' is at Q. We denote vectors by lowercase bold letters: $\mathbf{v}, \mathbf{w}$, etc. A vector $\mathbf{v}$ is also given as a pair of real numbers, for example $\mathbf{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$. (Notice that we use ( ) for vectors and not [ ] as for points.) The vector $\mathbf{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ is to be thought

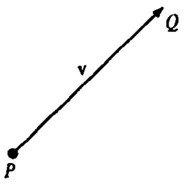

Figure 1.3

of as that displacement which carries the point $\begin{bmatrix}1\\3\end{bmatrix}$ into $\begin{bmatrix}6\\5\end{bmatrix}$, carries the point $\begin{bmatrix}-3\\2\end{bmatrix}$ into $\begin{bmatrix}2\\4\end{bmatrix}$ and, in general, carries any point $P=\begin{bmatrix}x\\y\end{bmatrix}$ into $Q=\begin{bmatrix}x+5\\y+2\end{bmatrix}$. Thus each vector **v** determines a (particular kind of) transformation of the affine plane into itself, a rigid translation of the whole plane. If P is any point in the plane, we will denote the displaced point Q by P"+"**v**: the "+" is a symbol for this operation of vectors on points. Thus **v** sends P into $Q = P$"+"**v**. Explicitly, if $P=\begin{bmatrix}x\\y\end{bmatrix}$ and $\mathbf{v}=\begin{pmatrix}a\\b\end{pmatrix}$, then P"+"$\mathbf{v}=\begin{bmatrix}x+a\\y+b\end{bmatrix}$.

We put quotation marks about the + sign because the operation is between two different kinds of object, points and vectors, and so differs from the usual notion of addition. Similarly, given any pair of points P and Q, there is a unique vector $\mathbf{v} = Q$"−"P such that

$$P\text{“}+\text{”}\mathbf{v} = Q.$$

We put quotation marks around the − because it relates different kinds of objects, it gives a vector from a pair of points. You should convince yourself, by working out some examples on graph paper, that two pairs of points, P, Q and R, S determine the same vector, i.e., Q"−"$P = S$"−"R, if and only if $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ are opposite sides of a parallelogram. For this reason, one frequently finds it said that a vector is determined by 'magnitude and direction'. But we want to refrain from introducing either magnitude or direction as they are not invariant concepts for us.

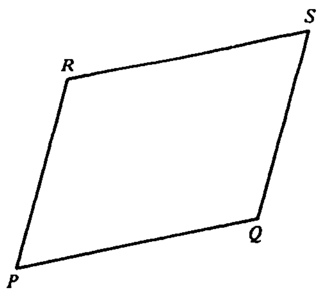

Figure 1.4

We can define the sum of two vectors: if $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} c \\ d \end{pmatrix}$, define their sum by

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} a + c \\ b + d \end{pmatrix}.$$

Notice that

$$P \text{``}+\text{''} (\mathbf{u} + \mathbf{v}) = (P \text{``}+\text{''} \mathbf{u}) \text{``}+\text{''} \mathbf{v} \tag{1.1}$$

since, if $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} c \\ d \end{pmatrix}$ and $P = \begin{bmatrix} x \\ y \end{bmatrix}$, then both the left and the right hand side of the above equation equal $\begin{bmatrix} a + c + x \\ b + d + y \end{bmatrix}$. The equation (1.1) says that the displacement corresponding to $\mathbf{u} + \mathbf{v}$ can be obtained by successively applying the displacement $\mathbf{v}$ and then the displacement $\mathbf{u}$. Notice that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. We can visualize the addition of vectors by the familiar parallelogram law: if we start with a point P and write $R = P \text{``}+\text{''} \mathbf{u}$, $Q = P \text{``}+\text{''} \mathbf{v}$ and $S = P \text{``}+\text{''} (\mathbf{u} + \mathbf{v})$, then the four points P, Q, S, R lie at the four vertices of a parallelogram. You should convince yourself of this fact by working out some examples on graph paper. The proof of this fact goes as follows. For any vector $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ and any real number t, define their product, $t\mathbf{v}$, by $t\mathbf{v} = \begin{pmatrix} ta \\ tb \end{pmatrix}$. If $\mathbf{v} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and P is any point, the set

$$l = \{P \text{``}+\text{''} t\mathbf{v}\} \quad (\text{as } t \text{ varies over } \mathbb{R})$$

is a straight line passing through P (just look at the third of our four descriptions of straight lines). If R is some other point, then the line

$$m = \{R \text{``}+\text{''} s\mathbf{v}\} \quad (\text{as } s \text{ varies over } \mathbb{R})$$

and l will intersect, i.e., have some point in common, if and only if there are some s_1 and t_1 such that

$$R \text{``}+\text{''} s_1 \mathbf{v} = P \text{``}+\text{''} t_1 \mathbf{v}$$

which means that

$$R = P \text{``}+\text{''} (t_1 - s_1)\mathbf{v}$$

and hence, for every s, that

$$R \text{``}+\text{''} s\mathbf{v} = P \text{``}+\text{''} (s + t_1 - s_1)\mathbf{v}.$$

This means that the lines m and l coincide. In other words, either the lines l and m coincide, or they do not intersect, i.e., either they are the same or they are parallel. Now let us go back to our diagram for vector addition. If $\mathbf{v} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then the point $Q = P \text{``}+\text{''} \mathbf{v}$ lies on the line $l$ through $P$ and the point $S = R \text{``}+\text{''} \mathbf{v}$ lies on the line m through R. There are now two possibilities: if the point R does *not* lie on the line l, so that $\mathbf{u} \neq t\mathbf{v}$ for any t, the lines l and m are parallel. A similar argument applies to the other two sides and we conclude that the figure is a parallelogram. If $\mathbf{u} = t\mathbf{v}$, then all four points lie on the line l. We can still view this picture as a sort of 'degenerate' parallelogram:

Figure 1.5

If either $\mathbf{u}$ or $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, the picture degenerates further:

Figure 1.6

We say that the vectors $\mathbf{u}$ and $\mathbf{v}$ are *linearly dependent* if there are numbers r and s, not both zero, such that

$$r\mathbf{u} + s\mathbf{v} = 0.$$

If $r \neq 0$ we can solve this equation for u to obtain $\mathbf{u} = -(s/r)\mathbf{v}$ and if $s \neq 0$ we can solve this equation for $\mathbf{v} = -(r/s)\mathbf{u}$. In either case, the 'addition parallelogram' degenerates into segments on a line $\left(\text{or if } \mathbf{u} = \mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ into a single point}\right)$. This is the reason for the term linearly dependent. If two vectors are not linearly dependent, we say that they are *linearly independent*.

The zero vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, denoted by $\mathbf{0}$, has the same point for its head and tail. It is called an *additive identity* because

$$\mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v} \quad \text{for all } \mathbf{v}.$$

The set of all vectors $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ where x and y are arbitrary real numbers is called $\mathbb{R}^2$. The space $\mathbb{R}^2$ is an example of a *vector space*, to be defined in the next section. The notational distinction between $\mathbb{R}^2$ and $A\mathbb{R}^2$ lies in the fact that in $\mathbb{R}^2$ the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has a special significance (it is the additive identity) and the addition of two vectors in $\mathbb{R}^2$ makes sense. These do not hold for $A\mathbb{R}^2$.

1.2. Vector spaces and their affine spaces

It is easy to check that the operations of addition of vectors in $\mathbb{R}^2$ and for multiplying vectors by real numbers satisfy the following collection of axioms:

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Laws for addition of vectors

Associative law of addition:	$(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
Commutative law of addition:	$\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
Existence of additive identity:	there is a vector $\mathbf{0}$ such that $\mathbf{0}+\mathbf{v}=\mathbf{v}$ for all $\mathbf{v}$.
Existence of additive inverse:	for every $\mathbf{v}$ there is a $-\mathbf{v}$ such that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$.

Laws involving the multiplication of vectors by real numbers

'One' acts as multiplicative identity:	$1\mathbf{v}=\mathbf{v}$ for every $\mathbf{v}$.
Associative and distributive laws:	for any real numbers r and s and any vectors $\mathbf{u}$ and $\mathbf{v}$

$$(rs)\mathbf{v}=r(s\mathbf{v})$$
$$(r+s)\mathbf{v}=r\mathbf{v}+s\mathbf{v}$$
$$r(\mathbf{u}+\mathbf{v})=r\mathbf{u}+r\mathbf{v}.$$

The above axioms are known as the axioms for a *vector space*. By definition, a vector space is a collection, V, of objects, $\mathbf{u}$, $\mathbf{v}$, etc., called vectors, such that we are given a binary operation, $+$, which assigns to every pair of vectors $\mathbf{u}$ and $\mathbf{v}$ a third vector $\mathbf{u}+\mathbf{v}$ and a multiplication which assigns to every real number t and every vector $\mathbf{v}$ another vector $t\mathbf{v}$ such that the above axioms hold.

We have verified that $\mathbb{R}^2$ is an example of a vector space. As a second example, we could take $\mathbb{R}^3$ where a vector now consists of a triplet

$$\mathbf{v}=\begin{pmatrix}a\\b\\c\end{pmatrix}$$

of real numbers. Addition of vectors is done componentwise as in $\mathbb{R}^2$:

$$\text{if } \mathbf{v}_1=\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix} \text{ and } \mathbf{v}_2=\begin{pmatrix}a_2\\b_2\\c_2\end{pmatrix}, \text{ then } \mathbf{v}_1+\mathbf{v}_2=\begin{pmatrix}a_1+a_2\\b_1+b_2\\c_1+c_2\end{pmatrix}.$$

The space $\mathbb{R}^3$ is just the space of vectors in our familiar three-dimensional space. We shall study the concept of dimension later on. We could also consider the space $\mathbb{R}=\mathbb{R}^1$ of the real numbers themselves as a vector space. Here addition is just the ordinary addition and multiplication ordinary multiplication. When we introduce the notion of dimension, this will be an example of a one-dimensional vector space.

As a different looking example of a vector space, consider the collection of all polynomials. We can add two polynomials:

$$(1+3x+7x^2)+(2-x^2+x^4-x^6)=3+3x+6x^2+x^4-x^6,$$

just add the coefficients. We can also multiply a polynomial by a real number:

$$7(1+3x+3x^2)=7+21x+21x^2.$$

You should check that the axioms for a vector space are satisfied. We can also consider the space of polynomials of at most a given degree. For example, the most general polynomial of degree at most two is of the form

$$P = ax^2 + bx + c.$$

The sum of two such polynomials

$$P_1 = a_1x^2 + b_1x + c_1 \quad \text{and} \quad P_2 = a_2x^2 + b_2x + c_2$$

is

$$P_1 + P_2 = (a_1 + a_2)x^2 + (b_1 + b_2)x + c_1 + c_2.$$

For example, if

$$P_1 = 3x^2 + 2x + 1, \quad P_2 = 7x^2 - 10x + 2$$

then

$$P_1 + P_2 = 10x^2 - 8x + 3.$$

The set of polynomials of degree at most two is also a vector space. Notice that it 'looks like' $\mathbb{R}^3$ in the sense that the preceding equations look like

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 7 \\ -10 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ -8 \\ 3 \end{pmatrix}.$$

We will return to this point later.

Suppose that we are *given* a vector space V; for example, V could be $\mathbb{R}^1$, $\mathbb{R}^2$ or $\mathbb{R}^3$. By an *affine space* associated to V, we mean a set A consisting of points P, Q, etc., and an operation "+" which assigns to each $P \in A$ and each $\mathbf{v} \in V$ another point in A which is denoted by P"+"$\mathbf{v}$. This rule is subject to the following axioms:

Associative law:	$(P``+"\mathbf{u})``+"\mathbf{v} = P``+"(\mathbf{u}+\mathbf{v})$ for any $P \in A$ and $\mathbf{u}, \mathbf{v} \in V$.
'Zero' acts as identity:	$P``+"\mathbf{0} = P$ for any $P \in A$.
Transitivity:	given any two points P and $Q \in A$, there is a $\mathbf{v} \in V$ such that $P``+"\mathbf{v} = Q$.
Faithfulness:	if, for any P, the equality $P``+"\mathbf{u} = P``+"\mathbf{v}$ holds, then $\mathbf{u} = \mathbf{v}$.

Combining the last two axioms, we can say that, given any two points P and Q, there is a *unique* vector $\mathbf{v}$ such that $P``+"\mathbf{v} = Q$. It is then sometimes convenient to write $\mathbf{v} = Q``-"P$.

The notion of a vector space and associated affine space lies at the basis of three centuries of physical thought, from Newtonian mechanics through special relativity and quantum mechanics. The purpose of the present chapter is to develop most of the key ideas in the study of these structures by examining the intuitively simple case of the two-dimensional* vector space $\mathbb{R}^2$. Let us begin, however, with some

* We will give a precise definition of the term 'two-dimensional' in §1.12, of 'one-dimensional' in a few lines, and of the general concept of the dimension of a vector space in Chapter 10.

comments about the one-dimensional case. Here the concepts are so 'obvious' that a detailed discussion of them may appear so pedantic as to be non-intuitive. Yet it is worth the effort.

A vector space V is called *one-dimensional* if it satisfies the following two conditions: (i) it possesses some vector $\mathbf{v} \neq \mathbf{0}$; and (ii) if $\mathbf{v} \neq \mathbf{0}$, then any $\mathbf{u} \in V$ can be written as $\mathbf{u} = r\mathbf{v}$ for some real number r. Notice that the r in this equation is unique: if

$$r_1\mathbf{v} = r_2\mathbf{v},$$

then we claim that $r_1 = r_2$. Indeed, from $r_1\mathbf{v} = r_2\mathbf{v}$ we can write

$$(r_1 - r_2)\mathbf{v} = \mathbf{0}.$$

If $r_1 - r_2 \neq 0$, then setting $s = (r_1 - r_2)^{-1}$, we have

$$\begin{aligned}\mathbf{0} = s[(r_1 - r_2)\mathbf{v}] &= (s(r_1 - r_2))\mathbf{v} \\ &= 1\mathbf{v} \\ &= \mathbf{v},\end{aligned}$$

so $\mathbf{v} = \mathbf{0}$, contradicting our original assumption that $\mathbf{v} \neq \mathbf{0}$. (You should check exactly which of the vector space axioms we used at each stage of the preceding argument.) Once we have chosen a $\mathbf{v} \neq \mathbf{0}$ in a one-dimensional vector space, then to each vector $\mathbf{u}$ there is assigned a real number, r,

$$\mathbf{u} \to r \quad \text{where} \quad \mathbf{u} = r\mathbf{v}.$$

If $\mathbf{u}_1 = r_1\mathbf{v}$ and $\mathbf{u}_2 = r_2\mathbf{v}$, then $\mathbf{u}_1 + \mathbf{u}_2 = (r_1 + r_2)\mathbf{v}$. Thus $\mathbf{u}_1 + \mathbf{u}_2$ corresponds to $r_1 + r_2$. Similarly, if $\mathbf{u} = r\mathbf{v}$ and t is any real number, then $t\mathbf{u} = (tr)\mathbf{v}$ so that $t\mathbf{u}$ corresponds to tr. In short, every vector corresponds to a real number, and the vector operations correspond to the operations on $\mathbb{R}^1$. We say that we have an *isomorphism* of the one-dimensional vector space V with $\mathbb{R}^1$. This identification of V with $\mathbb{R}^1$ depends on the choice of $\mathbf{v}$. A choice of $\mathbf{v}$ is called a choice of *basis* of V, and the number r associated to u via $\mathbf{u} = r\mathbf{v}$ is called the coordinate of $\mathbf{u}$ relative to the basis $\mathbf{v}$. Suppose we choose a different basis, $\mathbf{v}'$. Here $\mathbf{v}' = a\mathbf{v}$ where a is some non-zero real number. If $\mathbf{u} = r\mathbf{v}$, then

$$\mathbf{u} = (ra^{-1})av$$

so

$$\mathbf{u} = r'\mathbf{v}' \quad \text{where} \quad r' = a^{-1}r.$$

Thus, changing the basis, by replacing $\mathbf{v}$ by $a\mathbf{v}$, has the effect of changing the coordinate of any vector by replacing the coordinate r of any vector by $a^{-1}r$. The choice of a basis in a one-dimensional vector space is much like the choice of a unit for some physical quantity. If we change our units of mass from kilograms to grams, an object that weighs 1.3 kilograms now weighs 1300 grams. The difference is that, for many familiar physical quantities, the measurement of any object is given by positive numbers (or zero) only. It usually makes no sense to say that something has negative volume or mass, etc. An exception is in the theory of electricity, where electric charge can be positive or negative. For instance, we might