PART I

VECTOR AND TENSOR ALGEBRA

Throughout this book:

(i) *Lightface* Latin and Greek letters generally denote **scalars**.
(ii) *Boldface lowercase* Latin and Greek letters generally denote **vectors**, but the letters **o**, **x**, **y**, and **z** are reserved for **points**.
(iii) *Boldface uppercase* Latin and Greek letters generally denote **tensors**, but the letters **X**, **Y**, and **Z** are reserved for **points**.

1 Vector Algebra

We assume that the reader has had a basic course in vector algebra and, therefore, in introducing this subject, we take a relaxed approach that does not begin with formal definitions of point and vector spaces.

Roughly speaking, a point $\mathbf{x}$ is a dot in space and a vector $\mathbf{v}$ is an arrow that may be placed anywhere in space. As an example from everyday life, on a street map of a city

$$\mathbf{x} \stackrel{\text{def}}{=} \text{the corner of 4th Street and 5th Avenue}$$

might describe a point on the map, and to describe a second point $\mathbf{y}$ one-quarter of a mile northeast of $\mathbf{x}$ one might let

$$\mathbf{v} \stackrel{\text{def}}{=} \text{the vector whose direction is northeast and whose length is one-quarter mile}$$

and write $\mathbf{y} = \mathbf{x} + \mathbf{v}$. Thus, it would seem that a reasonable definition of a point space would require two basic notions: that of a point and that of a vector. Of course, granted a choice $\mathbf{o}$ of origin, one can identify all points with their vectors from $\mathbf{o}$; but such an identification is artificial, since there is no intrinsic way of defining $\mathbf{o}$. (What point on a street map would you call the origin?)

The space under consideration will always be a **three-dimensional Euclidean point space** $\mathcal{E}$. The term **point** will be reserved for elements of $\mathcal{E}$ and the term **vector** for elements of the associated vector space $\mathcal{V}$. Then:

(i) The difference $\mathbf{v} = \mathbf{y} - \mathbf{x}$ between the points $\mathbf{y}$ and $\mathbf{x}$ is a vector.
(ii) The sum $\mathbf{y} = \mathbf{x} + \mathbf{v}$ of a point $\mathbf{x}$ and a vector $\mathbf{v}$ is a point.
(iii) Unlike the sum of two vectors, the sum of two points has *no* meaning.

1.1 Inner Product. Cross Product

Our assumption that the point space $\mathcal{E}$ be Euclidean automatically endows the associated vector space $\mathcal{V}$ with an inner product.[3] We use the standard notation of vector analysis. In particular,

- The **inner product** (a scalar) and **cross product** (a vector)[4] of vectors $\mathbf{u}$ and $\mathbf{v}$ are respectively designated by

$$\mathbf{u} \cdot \mathbf{v} \qquad \text{and} \qquad \mathbf{u} \times \mathbf{v}.$$

[3] The inner product is often referred to as the dot product.
[4] We assume that the reader has some familiarity with these notions. The cross product is ordered in the sense that the cross product $\mathbf{u} \times \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is not generally equal to the cross product $\mathbf{v} \times \mathbf{u}$ of $\mathbf{v}$ and $\mathbf{u}$.

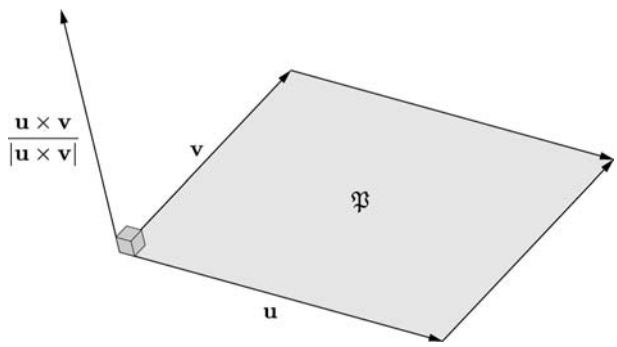

Figure 1.1. The parallelogram $\mathfrak{P}$ defined by the vectors $\mathbf{u}$ and $\mathbf{v}$ and the direction of $\mathbf{u}\times\mathbf{v}$ determined by the right-hand screw rule.

The inner product determines the magnitude (or length) of a vector $\mathbf{u}$ via the relation

$$|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}};$$

and the **angle**

$$\theta = \angle(\mathbf{u}, \mathbf{v}) \tag{1.1}$$

between nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \qquad (0 \le \theta \le \pi). \tag{1.2}$$

Since $-|\mathbf{u}||\mathbf{v}| \le \mathbf{u}\cdot\mathbf{v} \le |\mathbf{u}||\mathbf{v}|$, this definition assigns exactly one angle θ to each pair of nonzero vectors $\mathbf{u}$ and $\mathbf{v}$. Trivially,

$$\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta;$$

this relation is often used to define the inner product.

With regard to the cross product, the magnitude

$$|\mathbf{u}\times\mathbf{v}| \tag{1.3}$$

represents the **area spanned** by the vectors $\mathbf{u}$ and $\mathbf{v}$; that is, the *area* of the parallelogram $\mathfrak{P}$ **defined by** these vectors as indicated in Figure 1.1; this area is nonzero if and only if $\mathbf{u}$ and $\mathbf{v}$ are *linearly independent*.[5] Further, and what is most important, if $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, then:

(i) the magnitude of $\mathbf{u}\times\mathbf{v}$ is given by

$$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta \qquad (0 < \theta < \pi);$$

(ii) the vector $\mathbf{u}\times\mathbf{v}$ is *orthogonal* to both $\mathbf{u}$ and $\mathbf{v}$ with *direction* given by the right-hand screw rule.[6]

[5] Vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly dependent if, for some choice of the scalars a, b, and c, *not all zero*,

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}.$$

A similar definition applies to a pair of vectors $\mathbf{u}$ and $\mathbf{v}$. Finally, a set of vectors is *linearly independent* if it is not linearly dependent.

[6] Asserting formally that a right-hand screw revolved from $\mathbf{u}$ to $\mathbf{v}$ will advance in its nut toward $\mathbf{u}\times\mathbf{v}$ (Figure 1.1); cf. Brand (1947, §16) and Jeffrey (2002, §2.3). A simple consequence of this is that $\mathbf{v}\times\mathbf{u} = -\mathbf{u}\times\mathbf{v}$ and thus, in particular, that the cross product of $\mathbf{u}$ and $\mathbf{v}$ vanishes whenever $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent (Footnotes 4 and 5).

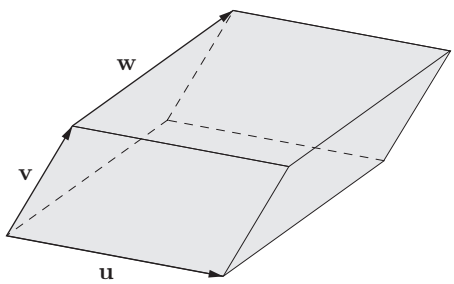

Figure 1.2. The parallelepiped defined by the vectors **u**, **v**, and **w**.

Since $\mathbf{u} \times \mathbf{v}$ gives the area of $\mathfrak{P}$, while $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathfrak{P}$, $\mathbf{u} \times \mathbf{v}$ is sometimes referred to as the **area vector** of $\mathfrak{P}$.

Similarly,

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \tag{1.4}$$

represents the **volume spanned** by the vectors **u**, **v**, and **w**; that is, the volume of the parallelepiped **defined by** these vectors as indicated in Figure 1.2. If this volume is nonzero, then **u**, **v**, and **w** are *linearly independent*.

If **u**, **v**, and **w** are linearly independent, then the **triad** $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ forms a **basis** for $\mathcal{V}$ in the sense that any vector **a** may be uniquely represented in terms of that triad, that is, there are unique scalars α, β, and γ such that

$$\mathbf{a} = \alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}.$$

A basis $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is **positively oriented**[7] if

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) > 0.$$

Two bases $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ have the **same orientation** if

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \ \text{ and } \ \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \ \text{ have the same sign.} \tag{1.5}$$

A basis $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is **orthonormal** if

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{u} = 0 \qquad \text{and} \qquad |\mathbf{u}| = |\mathbf{v}| = |\mathbf{w}| = 1, \tag{1.6}$$

so that **u**, **v**, and **w** are mutually orthogonal and of unit length.

A standard method of showing that two vectors **a** and **b** are equal uses the following result:

$$\mathbf{a} \cdot \mathbf{v} = \mathbf{b} \cdot \mathbf{v} \ \text{ for all vectors } \ \mathbf{v} \ \text{ if and only if } \ \mathbf{a} = \mathbf{b}. \tag{1.7}$$

The verification of this result is not difficult. Assume that $\mathbf{a} \cdot \mathbf{v} = \mathbf{b} \cdot \mathbf{v}$ for all **v**. Then the choice $\mathbf{v} = \mathbf{a} - \mathbf{b}$ yields $|\mathbf{a} - \mathbf{b}|^2 = 0$ and, hence, $\mathbf{a} = \mathbf{b}$. Similarly,

$$\mathbf{a} \times \mathbf{v} = \mathbf{b} \times \mathbf{v} \ \text{ for all vectors } \ \mathbf{v} \ \text{ if and only if } \ \mathbf{a} = \mathbf{b}, \tag{1.8}$$

which is a result whose proof we leave as an exercise.

A subset $\mathcal{K}$ of vectors is referred to as a **subspace** if, given any vectors **u** and **v** belonging to $\mathcal{K}$ and any scalars α and β, the **linear combination**

$$\alpha\mathbf{u} + \beta\mathbf{v} \ \text{ belongs to } \ \mathcal{K}. \tag{1.9}$$

Examples of subspaces of $\mathcal{V}$ are the singleton $\{\mathbf{0}\}$, a line through the origin, a plane through the origin, and $\mathcal{V}$ itself. There are no other examples.

[7] We view a positively oriented basis as right-handed, since the definition of the cross product is based on the right-hand screw rule (Footnote 6).

EXERCISE

1. Verify (1.8).

1.2 Cartesian Coordinate Frames

Throughout this book, *lowercase Latin subscripts range over the subset of integers*

$$\{1, 2, 3\}.$$

A **Cartesian coordinate frame** for $\mathcal{E}$ consists of a reference point $\mathbf{o}$ called **the origin** together with a positively oriented orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for $\mathcal{V}$. Being positively oriented and orthonormal, the basis vectors obey

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \qquad \text{and} \qquad \mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k) = \epsilon_{ijk}. \tag{1.10}$$

Here δ_{ij}, the Kronecker delta, is defined by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \tag{1.11}$$

while ϵ_{ijk}, the alternating symbol, is defined by

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } \{i, j, k\} = \{1, 2, 3\}, \{2, 3, 1\}, \text{ or } \{3, 1, 2\}, \\ -1, & \text{if } \{i, j, k\} = \{2, 1, 3\}, \{1, 3, 2\}, \text{ or } \{3, 2, 1\}, \\ 0, & \text{if an index is repeated}, \end{cases} \tag{1.12}$$

and, hence, has the value $+1$, -1, or 0 according to whether $\{i, j, k\}$ is an even permutation, an odd permutation, or not a permutation of $\{1, 2, 3\}$.

For brevity,

$$\{\mathbf{e}_i\} \stackrel{\text{def}}{=} \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$$

denotes a positively oriented orthonormal basis.

1.3 Summation Convention. Components of a Vector and a Point

Throughout this book, we employ the Einstein *summation convention* according to which summation over the range $1, 2, 3$ is implied for any index that is repeated twice in any term, so that, for instance,

$$u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3,$$

$$S_{ij} u_j = S_{i1} u_1 + S_{i2} u_2 + S_{i3} u_3,$$

$$S_{ik} T_{kj} = S_{i1} T_{1j} + S_{i2} T_{2j} + S_{i3} T_{3j}.$$

In the expression $S_{ij} u_j$, the subscript i is *free*, because it is not summed over, while j is a *dummy* subscript, since

$$S_{ij} u_j = S_{ik} u_k = S_{im} u_m.$$

When an expression in which an index is repeated twice *but summation is not to be performed* we state so explicitly. For example,

$$u_i v_i \qquad \text{(no sum)}$$

signifies that the subscript i is not to be summed over.

Next, if $S_{jk} = S_{kj}$, then

$$\begin{aligned}\epsilon_{ijk}S_{jk} &= \epsilon_{ikj}S_{kj}\\ &= -\epsilon_{ijk}S_{kj}\\ &= -\epsilon_{ijk}S_{jk}\end{aligned}$$

and vice versa; therefore

$$S_{ij} = S_{ji} \qquad \text{if and only if} \qquad \epsilon_{ijk}S_{jk} = 0. \tag{1.13}$$

Because $\{\mathbf{e}_i\}$ is a basis, every vector $\mathbf{u}$ admits the unique expansion

$$\mathbf{u} = u_j\mathbf{e}_j; \tag{1.14}$$

the scalars u_i are called the (Cartesian) **components of u** (relative to this basis). If we take the inner product of (1.14) with $\mathbf{e}_i$, we find that, since $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$,

$$u_i = \mathbf{u} \cdot \mathbf{e}_i. \tag{1.15}$$

Guided by this relation, we define the coordinates of a point $\mathbf{x}$ with respect to the origin $\mathbf{o}$ by

$$x_i = (\mathbf{x} - \mathbf{o}) \cdot \mathbf{e}_i. \tag{1.16}$$

In view of (1.14), the inner and cross products of vectors $\mathbf{u}$ and $\mathbf{v}$ may be expressed as

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= (u_i\mathbf{e}_i) \cdot (v_j\mathbf{e}_j)\\ &= u_i v_j \delta_{ij}\\ &= u_i v_i\end{aligned} \tag{1.17}$$

and

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_j\mathbf{e}_j) \times (v_k\mathbf{e}_k)\\ &= u_j v_k \mathbf{e}_j \times \mathbf{e}_k\\ &= \epsilon_{ijk}u_j v_k \mathbf{e}_i.\end{aligned} \tag{1.18}$$

In particular, (1.18) implies that the vector $\mathbf{u} \times \mathbf{v}$ has the component form

$$(\mathbf{u} \times \mathbf{v})_i = \epsilon_{ijk}u_j v_k. \tag{1.19}$$

When working with the cross product, the epsilon-delta identity

$$\epsilon_{ijk}\epsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}, \tag{1.20}$$

and its consequences

$$\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl} \qquad \text{and} \qquad \epsilon_{ijk}\epsilon_{ijk} = 6, \tag{1.21}$$

are useful. Also useful is the identity

$$\mathbf{e}_i = \tfrac{1}{2}\epsilon_{ijk}\mathbf{e}_j \times \mathbf{e}_k. \tag{1.22}$$

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors. Useful relations involving the inner and cross products then include

$$\begin{gathered}\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{v}\cdot(\mathbf{w}\times\mathbf{u}) = \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}),\\ \mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = 0,\\ \mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u},\\ \mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}.\end{gathered} \tag{1.23}$$

EXERCISES

1. Verify (1.22) and (1.23).
2. Establish the following identities for any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$:

$$(\mathbf{u}\cdot\mathbf{v})^2 + |\mathbf{u}\times\mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2,$$

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) + \mathbf{v}\times(\mathbf{w}\times\mathbf{u}) + \mathbf{w}\times(\mathbf{u}\times\mathbf{v}) = \mathbf{0}.$$

2 Tensor Algebra

2.1 What Is a Tensor?

We use the term **tensor** as a synonym for the phrase "linear transformation from $\mathcal{V}$ into $\mathcal{V}$." A tensor $\mathbf{S}$ is therefore a *linear* mapping of vectors to vectors; that is, given a vector $\mathbf{u}$,

$$\mathbf{v} = \mathbf{S}\mathbf{u} \tag{2.1}$$

is also a vector. One might think of a tensor $\mathbf{S}$ as a machine with an input and an output: if a vector $\mathbf{u}$ is the input, then the vector $\mathbf{v} = \mathbf{S}\mathbf{u}$ is the output (Figure 2.1). The **linearity** of a tensor $\mathbf{S}$ is embodied by the requirements:

$$\mathbf{S}(\mathbf{u}+\mathbf{v}) = \mathbf{S}\mathbf{u} + \mathbf{S}\mathbf{v} \qquad \text{for all vectors } \mathbf{u} \text{ and } \mathbf{v};$$

$$\mathbf{S}(\alpha\mathbf{u}) = \alpha\mathbf{S}\mathbf{u} \qquad \text{for all vectors } \mathbf{u} \text{ and scalars } \alpha.$$

Tensors $\mathbf{S}$ and $\mathbf{T}$ are equal if their outputs are the same whenever their inputs are equal; precisely,

$$\mathbf{S} = \mathbf{T} \ \text{ if and only if } \ \mathbf{S}\mathbf{v} = \mathbf{T}\mathbf{v} \ \text{ for all vectors } \ \mathbf{v}. \tag{2.2}$$

One way of showing that tensors $\mathbf{S}$ and $\mathbf{T}$ are equal is a conseqence of the following result:

$$\mathbf{a}\cdot\mathbf{S}\mathbf{b} = \mathbf{a}\cdot\mathbf{T}\mathbf{b} \ \text{ for all vectors } \ \mathbf{a} \ \text{ and } \ \mathbf{b} \ \text{ if and only if } \ \mathbf{S} = \mathbf{T}. \tag{2.3}$$

To prove this, we write $\mathbf{a}\cdot\mathbf{S}\mathbf{b} = \mathbf{a}\cdot\mathbf{T}\mathbf{b}$ in the form $\mathbf{a}\cdot(\mathbf{S}\mathbf{b} - \mathbf{T}\mathbf{b})$, which, by (1.7), holds for all $\mathbf{a}$ if and only if $\mathbf{S}\mathbf{b} = \mathbf{T}\mathbf{b}$ and the validity of this for all $\mathbf{b}$ yields, by (2.2), $\mathbf{S} = \mathbf{T}$.

Consistent with (2.2), tensors are generally defined by their actions on arbitrary vectors. For example, the sum $\mathbf{S}+\mathbf{T}$ of tensors $\mathbf{S}$ and $\mathbf{T}$ and the product $\alpha\mathbf{S}$ of a tensor $\mathbf{S}$ and a scalar α are defined as follows:

$$\begin{aligned}(\mathbf{S}+\mathbf{T})\mathbf{v} &= \mathbf{S}\mathbf{v} + \mathbf{T}\mathbf{v},\\ (\alpha\mathbf{S})\mathbf{v} &= \alpha(\mathbf{S}\mathbf{v}),\end{aligned} \tag{2.4}$$

for all vectors $\mathbf{v}$. As a consequence of these definitions, the set of all tensors forms a vector space (of dimension 9).

EXERCISE

1. Show that $\mathbf{S}+\mathbf{T}$ and $\alpha\mathbf{S}$ defined by (2.4) are actually *linear* and hence tensors.

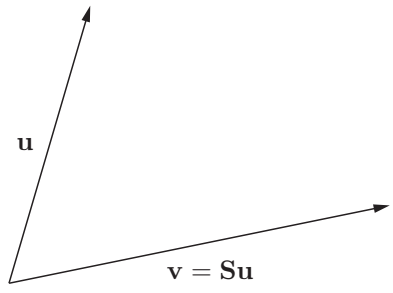

Figure 2.1. A vector $\mathbf{u}$ and its transformation $\mathbf{v} = \mathbf{S}\mathbf{u}$ by a tensor $\mathbf{S}$.

2.2 Zero and Identity Tensors. Tensor Product of Two Vectors. Projection Tensor. Spherical Tensor

Two basic tensors are the **zero tensor** $\mathbf{0}$ and the **identity tensor** $\mathbf{1}$, defined by

$$\mathbf{0}\mathbf{v} = \mathbf{0} \qquad \text{and} \qquad \mathbf{1}\mathbf{v} = \mathbf{v}$$

for all vectors $\mathbf{v}$.

Another example of a tensor is the **tensor product** $\mathbf{u} \otimes \mathbf{v}$, of two vectors $\mathbf{u}$ and $\mathbf{v}$, defined by

$$(\mathbf{u} \otimes \mathbf{v})\mathbf{w} = (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \tag{2.5}$$

for all $\mathbf{w}$. By (2.5), the tensor $\mathbf{u} \otimes \mathbf{v}$ maps any vector $\mathbf{w}$ onto a scalar multiple of $\mathbf{u}$. Let $\mathbf{e}$ be a unit vector. Then, since

$$(\mathbf{e} \otimes \mathbf{e})\mathbf{u} = (\mathbf{u} \cdot \mathbf{e})\mathbf{e}$$

for any vector $\mathbf{u}$, the tensor $\mathbf{e} \otimes \mathbf{e}$ maps each vector $\mathbf{u}$ to the projection $(\mathbf{u} \cdot \mathbf{e})\mathbf{e}$ of $\mathbf{u}$ onto the vector $\mathbf{e}$. Similarly,

$$(\mathbf{1} - \mathbf{e} \otimes \mathbf{e})\mathbf{u} = \mathbf{u} - (\mathbf{u} \cdot \mathbf{e})\mathbf{e},$$

so that, for any vector $\mathbf{u}$, the tensor $\mathbf{1} - \mathbf{e} \otimes \mathbf{e}$ maps each vector $\mathbf{u}$ to the projection $\mathbf{u} - (\mathbf{u} \cdot \mathbf{e})\mathbf{e}$ of $\mathbf{u}$ onto the plane perpendicular to $\mathbf{e}$. The tensors

$$\mathbf{e} \otimes \mathbf{e} \qquad \text{and} \qquad \mathbf{1} - \mathbf{e} \otimes \mathbf{e} \tag{2.6}$$

therefore define **projections** onto $\mathbf{e}$ and onto the plane perpendicular to $\mathbf{e}$ (Figure 2.2).

Next, note that

$$\begin{aligned} \mathbf{1}\mathbf{v} &= \mathbf{v} \\ &= (\mathbf{v} \cdot \mathbf{e}_i)\mathbf{e}_i \\ &= (\mathbf{e}_i \otimes \mathbf{e}_i)\mathbf{v} \end{aligned}$$

for all $\mathbf{v}$; thus, by (2.2), we have the useful identity

$$\mathbf{1} = \mathbf{e}_i \otimes \mathbf{e}_i. \tag{2.7}$$

Finally, a tensor $\mathbf{S}$ of the form

$$\mathbf{S} = \alpha\mathbf{1}, \tag{2.8}$$

with α a scalar, is called a **spherical tensor**.

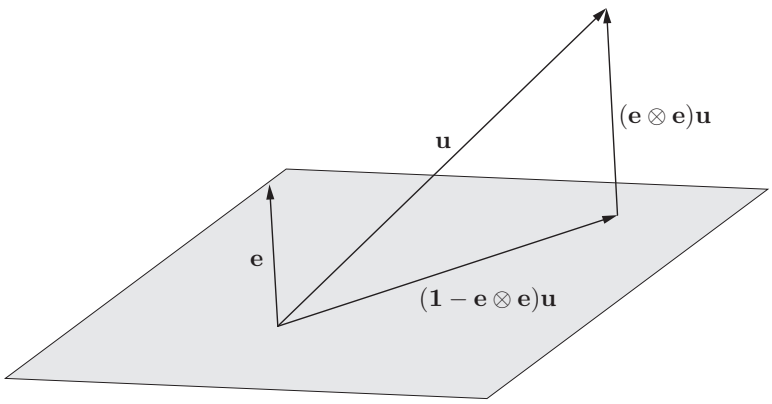

Figure 2.2. The projections $(\mathbf{e}\otimes\mathbf{e})\mathbf{u} = (\mathbf{u}\cdot\mathbf{e})\mathbf{e}$ and $(\mathbf{1}-\mathbf{e}\otimes\mathbf{e})\mathbf{u} = \mathbf{u}-(\mathbf{u}\cdot\mathbf{e})\mathbf{e}$ of vector $\mathbf{u}$ onto a unit vector $\mathbf{e}$ and onto the plane perpendicular to that unit vector.

EXERCISE

1. Show that, for $\mathbf{a}$ and $\mathbf{b}$ nonzero,

$$\mathbf{a}\otimes\mathbf{b} = \mathbf{c}\otimes\mathbf{d} \qquad \text{if and only if } \mathbf{c} = \alpha\mathbf{a} \text{ and } \mathbf{d} = \beta\mathbf{b} \text{ with } \alpha\beta = 1.$$

2.3 Components of a Tensor

Given a tensor $\mathbf{S}$, choose an arbitrary vector $\mathbf{u}$ and let

$$\mathbf{v} = \mathbf{S}\mathbf{u}.$$

Then,

$$\begin{aligned} v_k\mathbf{e}_k &= \mathbf{S}(u_j\mathbf{e}_j) \\ &= u_j\mathbf{S}\mathbf{e}_j \end{aligned}$$

and, by (1.10), taking the inner product of this relation with $\mathbf{e}_i$ yields

$$v_i = (\mathbf{e}_i\cdot\mathbf{S}\mathbf{e}_j)u_j;$$

thus, defining the **components** of $\mathbf{S}$ by

$$\begin{aligned} S_{ij} &= (\mathbf{S})_{ij} \\ &= \mathbf{e}_i\cdot\mathbf{S}\mathbf{e}_j, \end{aligned} \tag{2.9}$$

we see that the component form of the relation $\mathbf{v} = \mathbf{S}\mathbf{u}$ is

$$v_i = S_{ij}u_j. \tag{2.10}$$

Further, this relation implies that

$$\begin{aligned} \mathbf{v} &= v_i\mathbf{e}_i \\ &= (S_{ij}u_j)\mathbf{e}_i \\ &= (S_{ij}\mathbf{e}_j\cdot\mathbf{u})\mathbf{e}_i \\ &= (S_{ij}\mathbf{e}_i\otimes\mathbf{e}_j)\mathbf{u}, \end{aligned}$$

and since $\mathbf{v} = \mathbf{S}\mathbf{u}$ we must have

$$\mathbf{S}\mathbf{u} = (S_{ij}\mathbf{e}_i\otimes\mathbf{e}_j)\mathbf{u}.$$