CAMBRIDGE TRACTS IN MATHEMATICS

General Editors
B. BOLLOBAS, H. HALBERSTAM, C. T. C. WALL

99 Some applications of modular forms
PETER SARNAK

Stanford University

Some applications of modular forms
To my parents Frieda and Leon
Preface

These notes are an expanded version of the Wittemore Lectures given at Yale in November 1988.¹ The material presented in the four chapters is more or less self-contained. On the other hand, in the section at the end of each chapter called 'Notes and comments,' it is assumed that the reader is familiar with more advanced and sophisticated notions from the theory of automorphic forms. Some of the material presented here overlaps with a forthcoming book, ‘Discrete groups, expanding graphs and invariant measures’ by A. Lubotzky. The points of view, emphasis, and presentation in that book and the present notes are sufficiently different that we decided to keep the two works separate. The reader is encouraged to look at both treatments of the material.

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