CAMBRIDGE TRACTS IN MATHEMATICS

101 Infinite electrical networks

This is the first book to present the salient features of the general theory of infinite electrical networks in a coherent exposition. Using the basic tools of functional analysis and graph theory, the author presents the fundamental developments of the past two decades and discusses applications to other areas of mathematics and engineering.

The jump in complexity from finite electrical networks to infinite ones is comparable to the jump in complexity from finite-dimensional to infinite-dimensional spaces. Many of the questions conventionally asked about finite networks are currently unanswerable for infinite networks, while questions that are meaningless for finite networks crop up for infinite ones and lead to surprising results, such as the occasional collapse of Kirchhoff's laws in infinite regimes. Some central concepts have no counterpart in the finite case, for example, the extremities of an infinite network, the perceptibility of infinity, and the connections at infinity.

The first half of the book presents existence and uniqueness theorems for both infinite-power and finite-power voltage-current regimes, and the second half discusses methods for solving problems in infinite cascades and grids. A notable feature is the recent invention of transfinite networks, roughly analogous to Cantor's extension of the natural numbers to the transfinite ordinals. The last chapter is a survey of applications to exterior problems of partial differential equations, random walks on infinite graphs, and networks of operators on Hilbert spaces.

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Preface

... accumulations of isolated facts and measurements which lie as a sort of dead weight on the scientific stomach, and which must remain undigested until theory supplies a more powerful solvent ... .

Lord Rayleigh

The theory of electrical networks became fully launched, it seems fair to say, when Gustav Kirchhoff published his voltage and current laws in 1847 [72]. Since then, a massive literature on electrical networks has accumulated, but almost all of it is devoted to finite networks. Infinite networks received scant attention, and what they did receive was devoted primarily to ladders, grids, and other infinite networks having periodic graphs and uniform element values. Only during the past two decades has a general theory for infinite electrical networks with unrestricted graphs and variable element values been developing. The simpler case of purely resistive networks possesses the larger body of results. Nonetheless, much has also been achieved with regard to RLC networks. Enough now exists in the research literature to warrant a book that gathers the salient features of the subject into a coherent exposition.

As might well be expected, the jump in complexity from finite electrical networks to infinite ones is comparable to the jump in complexity from finite-dimensional spaces to infinite-dimensional spaces. Many of the questions we conventionally ask and answer about finite networks are unanswerable for infinite networks – at least at the present time. On the other hand, questions, which are meaningless for finite networks, crop up about infinite ones and lead to novel attributes, which often jar the habits of thought conditioned by finite networks. A case in point is the occasional collapse of such fundamentals as Kirchhoff's laws. Indeed, Kirchhoff's current law need not hold at a node with an infinity of incident branches, and his voltage law may fail around an infinite loop. Moreover, the use of an infinite loop is itself an issue, and different voltage-current regimes can arise depending on whether or not infinite loops are allowed. The theory of infinite electrical networks is perforce very different from that of finite networks. Concepts that have no counterparts in the finite case are fundamental to the subject, for example, the extremities of an infinite network, the perceptibility of infinity, the restraining or nonrestraining character of an infinite node, and the connections at infinity. This perhaps is one of the reasons why it took so long for the subject to become a distinctive research area.

But why bother, one may ask, to examine it at all? One answer is that it is intellectually challenging. Homo sapiens has the ability to use his/her mind in cognitive endeavors that do not directly relate to his/her physical needs – and does so. Here is one more arena in which to exercise that proficiency. To put this another way, circuit theory is intrinsically a mathematical discipline and its open problems will be attacked.

Another answer is that the subject does have practical applications. For example, a variety of partial differential equations such as Poisson's equation, the heat equation, the acoustic wave equation, and polarized forms of Maxwell's equations, have partial difference approximations that are realized by electrical networks. Moreover, the domains in which those equations are to hold are at times appropriately viewed as being infinite in extent, for instance, when a fringing problem is at hand or when a wave is propagating into an exterior region. In these cases, the discretized models are infinite electrical networks. To be sure, one may truncate the domain to obtain a finite network, but a search for a solution to the infinite network would avoid the additional error imposed by the truncation. It should be admitted, however, that much of infinite network theory does not possess engineering or physical significance – as yet. Nonetheless, though the intellectual challenge of infinite networks is an attraction, it is not only curiosity that leads one into circuit abstractions. It is conducive to the growth of that discipline to pursue, at least by some of us, circuit theory wherever it may lead.

Another explanation for the relative inattention paid to this stimulating subject is its apparent isolation from the established disciplines. It seems to be too much like electrical engineering to attract mathematicians and too much like mathematics to attract engineers. Nonetheless, it has been developing apace in recent years, and it is accessible to anyone with some knowledge of circuit theory and elements of functional analysis. Moreover, the subject is in its puberty; there are many open problems and undoubtedly much still to be discovered. Theoreticians looking for new research horizons may well consider infinite electrical networks.

We assume that the reader has a basic knowledge of graph theory and functional analysis. Nonetheless, we explicitly state the principal results from these subjects that we use and provide references to textbooks for expositions of them. A summary of most of the needed standard results from functional analysis is given in the appendices of [154]. On the other hand, since an electrical network consists of a certain analytical structure imposed upon a graph, the graphs we encounter in this book are almost entirely infinite ones. We will restrict our attention exclusively to countable graphs, that is, to those having finite or denumerable sets of branches. Thus, the graph theory we shall employ is not the standard theory encountered in the customary courses on finite circuits. Finally, it would be helpful to the reader to have some prior exposure to electrical circuit theory; such knowledge would enhance a comprehension of this book's subject matter, but is not essential because we shall define and derive all the concepts and results in circuit theory that we use. Actually, this book is addressed to two different audiences: engineers and mathematicians. Hence, background information that is well-known to one group may be belabored for the sake of the other. Moreover, we strive for a rigorous exposition and, on the other hand, present many examples to illuminate this radical extension of circuit theory.

The topics are organized as follows. The first chapter starts with some fundamental concepts and definitions. It continues with a variety of examples, which illustrate the peculiarities and paradoxes that distinguish infinite networks from finite ones. It then presents two examples to illustrate how infinite networks arise as discretizations of physical phenomena in exterior domains. It ends by pointing out how the transient behavior of a linear RLC network can be obtained from the analysis of a resistive network.

There is a dichotomy in the theory of infinite networks, which arises from the fact that Ohm's law and Kirchhoff's voltage and current laws do not by themselves determine a unique voltage-current regime except in certain trivial cases. This leads to two divergent ways of studying infinite electrical networks. One way is to impose those laws alone and to examine the whole class of different voltage-current regimes that the network can have. This is done in Chapter 2. In general, the power dissipated in the network under such a regime is infinite. Chapter 2 is not essential to the rest of the book and can be skipped if one is interested only in finite-power regimes.

The second way, which is examined in Chapter 3 for linear networks and in Chapter 4 for nonlinear ones, is motivated by the following question. What additional conditions must be added to Ohm's law and Kirchhoff's laws to ensure a unique voltage-current regime? One conspicuous requirement that suggests itself is finiteness of the total dissipated power. This suffices for some networks but not for all. In general, what is occurring at infinity has to be specified as well. For certain networks infinity is imperceptible from any particular point of the network, and so the connections at infinity can remain unspecified, but in other cases it is necessary to know what is connected to the network at infinity if a unique voltage-current regime is to be obtained.

In the latter case, what is connected out at infinity may be another infinite network. Thus, we are led naturally to the idea of a transfinite network, one wherein two nodes may be connected through a transfinite path but not through a finite one. Actually, this process of constructing a network that "extends beyond infinity" can be repeated by connecting infinite collections of infinite networks at their extremities to obtain hierarchies of transfinite networks. Roughly speaking, we might say that the process is analogous to the construction of an ordinary infinite network by connecting an infinity of branches at various nodes; in the transfinite case, branches are replaced by infinite networks and the nodes of a branch are replaced by the extremities of an infinite network. This generalization of infinite networks is explored in Chapter 5.

Chapters 2 through 5 are concerned almost entirely with existence and uniqueness theorems. To be sure, some of the proofs therein are constructive so that solutions can in principle be calculated. Nevertheless, the focus of those chapters is not on the computability of solutions. If, however, we restrict our attention to infinite networks having certain regularities, new methods become available, which in some cases lead to ways of calculating voltage-current regimes. A particularly simple regularity is the one-dimensional infinite cascade of three-terminal and two-port networks. This is the subject of Chapter 6. Chapter 7 considers various kinds of grounded and ungrounded grids. A useful application arises in this context. Since the discretizations of various partial differential equations can often be realized by electrical grids, methods for solving infinite electrical networks may be used to compute solutions for the differential equations for which the the domain of analysis is infinite in extent. In fact, such procedures may avoid the domain truncations customarily used to convert the infinite domain into an approximating finite one. Applications of this nature are surveyed in Sections 8.1 and 8.2.

It should be emphasized, however, that this book is not a comprehensive treatment of all aspects of infinite electrical networks. For instance, its thrust is toward resistive networks. To be sure, several of our analyses extend to RLC networks, and we point this out as occasions arise; but RLC networks are discussed only in this peripheral way. Another important issue is the growth of a network, something we refer to as the "perceptibility of infinity." It relates to some fundamental problems concerning random walks, Markov chains, and the classification of Riemann surfaces. A substantial body of results in this regard has been accumulating in recent years. This is very briefly covered in Section 8.3, wherein a number of references to its literature are listed. Still another context in which infinite electrical networks naturally arise is the theory of operator networks, that is, finite or infinite networks whose parameters are Hilbert-space operators. One case of this will be met in Chapter 7 wherein grids are decomposed into ladder networks of Laurent operators. Operator networks of greater generality are briefly discussed in Section 8.4, the last section.

In short, this book might be described figuratively as "roadbuilding through the jungle of infinite electrical networks along the valley of resistive networks."

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